REPRESENTING GENERALIZED CYLINDERS

Ambjörn Naeve and Jan-Olof Eklundh
Computational Vision and Active Perception Laboratory
Department of Numerical Analysis and Computing Science
Royal Institute of Technology, S-100 44 Stockholm, Sweden

1 Introduction

In an influential paper Binford 1971 [1] proposed generalized cylinders as a model for representing visual shape. With reference to Blum´s medial axis transform he argued that objects, or at least their parts, often can be considered as generated by sweeping a planar curve, the cross section, along an axis or a spine. For instance, both manufactured objects and objects that accrue from natural growth or wearing tend to have such shapes. Extensive work on representation and understanding of shape in computer vision has since been based on this notion, and it has also been used to form theories for object recognition by humans [2]. Despite these vast efforts on applying the generalized cylinders to problems on shape, the general representation that Binford suggested does not necessarily define non-singular surfaces, i.e. surfaces that are boundaries of physical objects. This problem has largely been addressed by only considering very special cases, the main one being that of straight homogeneous generalized cylinders (SHGC:s). Moreover, as pointed out by Koenderink ([3], p. 593) the GC notion is so general that it contains almost everything. Of course, one can argue that it is more natural to consider volumetric representations than surface representations for physical objects. Nevertheless, it is undisputable that surfaces can be observed visually. It is also true that the intuition behind Binford´s suggested model is quite natural. The attempts to either restrict the model to oversimplified cases, or to only consider local properties is no doubt critical for solving certain recognition problems, but they neither fully exploit the global aspect, nor the intuitiveness of Binford´s definition. Therefore, it must be of interest to ask if there is any more precise definition of generalized cylinders that specifies the notion better.

In this paper we will address this question from two different perspectives. First, we will consider the general definition of the generalized cylinder surface class and from it derive conditions that guarantee a non-singular surface. Secondly, we will consider some classes of surfaces that obviously fall within the original definition, but are more general than the SHGC. Here we will not discuss the problem that presumably has caused the development into mainly studying SHGC:s namely that of what properties can be observed from an image, i.e. invariants in a general sense. In the end this is of course the underlying question that must guide the development towards a better taxonomy that we are attempting. There is a wealth of such results, in particular concerning SHGC:s (see e.g. [4]-[9]). However, we feel that the basic geometry of the situation is worth to analyze on its own account, and this is the topic of the present paper.

The taxonomy that we will discuss here can be considered to evolve either in a top-down or a bottom-up direction. It is introduced in a *top-down* manner, regarding the shape of a surface as evolving dynamically from the general towards the special by applying a series of successive specializations, introduced in the form of a hierarchy of *tightenings* of parametric constraints. This process is analogous to the manufacturing of a special kind of surface from a general piece of material by the successive application of various tools - e.g. lathing a rotationally symmetric surface from a non-symmetric piece of wood - each tool leaving its mark of symmetry on the shape of the material. However, our taxonomy can also be "applied in reverse", i.e. in a *bottom-up* way, starting from a specialized shape and subjecting it to a series of successive generalizations, corresponding to a sequence of *relaxations* of the parametric constraints. This is analogous to manufacturing a surface from a highly specialized type of shape and distorting its initial symmetry in various ways. As an example, a plastic cylinder can be heated and vaccuumsuctioned into a rotationally symmetric surface, which can then be bent in such a way that its axis is formed into e.g. a circle (Fig. 3). The resulting surface is neither cylindrical, nor rotationally symmetric, but it will still qualify as a natural subtype of generalized cylinder, whose constructional history can be quantitatively described in terms of constraint relaxations of its shape parameters. In fact, this bottom-up view is more closely related to the process of shape perception, where one initially matches simple shapes against the unknown object in order to determine e.g. the best "cylindrical fit", and then introduces various deviations from the cylindrical shape in order to increase the closeness of the fit.

2 General Parametrization and Condition of Regularity

Our framework is based on a general parametrization method for generalized cylinders that formalizes Binford´s intuitive idea. We will use this parametrization to formulate a general regularity condition as well as to suggest a taxonomy scheme for such surfaces.

A *generalized cylinder* S (in the sense of Binford's) is generated as a point-locus of a planar curve C (the *cross-section curve*) that is subjected to some motion M in space and allowed to change its shape during this motion. Let the plane of C be called π and let any one of the two unit normals of π be denoted by $\mathbf{u}$. Consider a fixed point O in π and choose a 2-dimensional cartesian coordinate system in π with the origin at O and unit base vectors $\mathbf{i}$ and $\mathbf{j}$. By tracking O during the motion M of π we associate with this motion a space-curve Γ called a *spine-curve* of S. Using the arclength s of Γ as parameter, the motion $M(s)$ can be decomposed into a translation $T(s)$ of the point O and a rotation $A(s)$ of the plane π around the point $T(s)O = O(s) = (x_0(s), y_0(s), z_0(s))$. If ξ and η denote the internal coordinates of the cross-section curve C in the system $\{\mathbf{i}, \mathbf{j}\}$ of π, the cross-section curve $C(s, t)$ is expressed relative to $O(s)$ by:

$$C(s,t) = \xi(s,t)\,\mathbf{i}(s) + \eta(s,t)\,\mathbf{j}(s) \quad . \tag{1}$$

In equation (1) t represents the internal parameter of C while s represents the external parameter of M. The dependence of ξ and η on s reflects the possibility of shape change of C during the motion. Hence we can express the points on the generated (parametrized) surface S in the following way:

$$S(s,t) = T(s)\,O + A(s)\left[\xi(s,t)\ \ \eta(s,t)\right] \quad , \tag{2}$$

or in coordinate form

$$\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} x_0(s) \\ y_0(s) \\ z_0(s) \end{bmatrix} + \begin{bmatrix} i_1(s) & j_1(s) & u_1(s) \\ i_2(s) & j_2(s) & u_2(s) \\ i_3(s) & j_3(s) & u_3(s) \end{bmatrix} \begin{bmatrix} \xi(s,t) \\ \eta(s,t) \\ 0 \end{bmatrix} \quad . \tag{3}$$

Here the columns of the 3x3 orthogonal matrix $A(s)$ are the "external" coordinates of the vectors $\mathbf{i}(s)$, $\mathbf{j}(s)$ and $\mathbf{u}(s)$ respectively. With $\mathbf{x} = (x, y, z)$, $\mathbf{r} = (x_0, y_0, z_0)$ and $\mathbf{c} = (\xi, \eta, 0)$ we can write (3):

$$\mathbf{x}(s,t) = \mathbf{r}(s) + A(s)\,\mathbf{c}(s,t) \quad . \tag{4}$$

It is clear that (4) represents a way to parametrize any form of GC. Without some kind of constraints on the spine-curve Γ and the cross-section C, this kind of parametrization is by no means unique. In fact, for any given Γ, any surface S can be regarded as a GC with spine Γ in an infinity of different ways. From (4) we can easily deduce a sufficient condition of regularity for the surface, namely:

$$\begin{aligned} \mathbf{x}_s(s,t) &= \mathbf{r}'(s) + A'(s)\,\mathbf{c}(s,t) \quad , \\ \mathbf{x}_t(s,t) &= A(s)\,\mathbf{c}_t(s,t) \quad , \end{aligned} \tag{5}$$

where

$$\mathbf{x}_s \times \mathbf{x}_t \neq 0 \quad , \tag{6}$$

which guarantees that the normal to *S* exists - and is parallel to $\mathbf{x}_s \times \mathbf{x}_t$ - for each point (s, t) that satisfies (6). Hence (5) and (6) constitute a way to verify that a given generalized cylinder parametrization defines a regular surface. Of course, the vanishing of $\mathbf{x}_s \times \mathbf{x}_t$ - does not necessarily imply the non-existence of the surface normal at the corresponding point. We will not pursue this matter further here, but note that Ponce [6] in fact gives a necessary and sufficient condition for the regularity of a SHGC.

3 Taxonomy

Having defined a general way to parametrize a generalized cylinder, and expressed the regularity condition of the surface relative to this parametrization, we will now use our parametrization to attempt to formulate a taxonomy for generalized cylinders. This will be done by introducing a hierarchy of constraints on the parametrization (4) that define parametric subclasses in the corresponding way. These constraints are chosen to include the traditional classification of generalized cylinders within the field of computer vision - such as e.g. SHGC, Piped surfaces and Rotational surfaces. We also introduce two natural steps of generalization of Rotational surfaces - that we term *Circular Sections Twist Compensated* surfaces (CSTC) respectively *Parallel Sections Twist Compensated* surfaces (PSTC). The surfaces of type CSTC relax the constraint of straight spine (= axis of rotation), and the surfaces of type PSTC relax as well the constraint of circular section curves for a Rotational surface. What survives in both of these cases is the orthogonal net corresponding to the curves of section and their orthogonal trajectories, whence generalizing the net of meridians and parallel circles in the case of a Rotational surface.

The constraint hierarchy that we introduce can be seen as a process of shape evolution by successive specializations. The underlying analogy is the application of machine tools to an initital object of generic shape *S*. A collection of shape subtypes of *S* could be imagined as a set of machine made surfaces, where each machine takes a surface of generic type (*S*) as input and leaves its mark of symmetry on the produced "output surface". For example, a lathe turns out a rotational surface, and laminating rollers produce a developable surface.The hierarchy of constraints is presented in the form of a so called *shape graph*, (Fig. 1) where each shape type (box) is considered as a specialization of the types connected to it from above. Most of the specializations are simply expressed by multiple inheritance (i.e. several paths leading down into the same shape subtype box), but sometimes an additional constaint is needed in order to arrive at the corresponding subtype. In such cases this additional constraint is expressed by a number (e.g. **5**. in Fig. 1) directly on top of the subtype to which the constraint applies. This number also appears in the text, together with an equation expressing the corresponding parametric constraint.

In this way a general parametrization of type (4) gives rize to a dynamic shape type system of subclasses of generalized cylinders, where each subtype is conceived as having evolved from a general prototype by a process of specialization (a "constructional history") - corresponding to a sequence of successive tightenings of parametric constraints. Naturally, any shape graph will always be incomplete - showing only the shape types and transitions that we want to consider for some reason or other. However, the shape graph is open to expansion. This is exemplified by the shape types of Fig. 1, denoted by “Plane Spine” respectively” Linear Sections”. These shape types are of course also subtypes of the general GC type, but they have been left unconnected to the top GC-type as a notational convenience. We now describe our shape graph of GC:s (see [10] for more details). From the general GC box of Fig. 1 we have introduced five different parametric constraints:

1. $\mathbf{r}'(s) = \text{constant vector}$ **(7)**

2. $A(s) = \text{identity matrix}$ **(8)**

3. $$\mathbf{c}(s,t) = \begin{bmatrix} f(s)\,g_1(t) \\ f(s)\,g_2(t) \\ 0 \end{bmatrix} \qquad (9)$$

4. $$\mathbf{c}(s,t) = ParallelCurve(Distance(s), \mathbf{c}(s_0, t)) \qquad (10)$$

5. $u(s) = k\mathbf{r}'(s) \quad , \qquad k = \text{constant}$ **(11)**

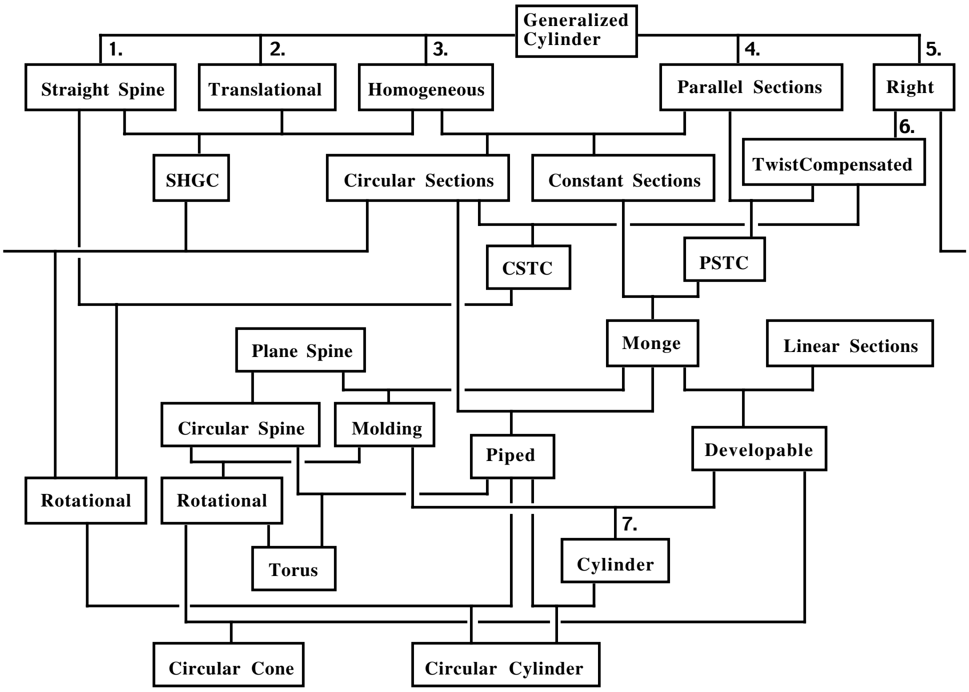

Fig. 1 A shape graph (with cylindrical connectivity) for the family of generalized cylinders.

leading downwards to five corresponding shape subtypes of GC. The numbers **1** - **5** correspond between the above formulas and Fig. 1. The constraint **1** leads to the type *Straight Spine GC* since this constraint obviously implies that the spine curve $\mathbf{r}(s)$ must be a straight line segment. If **2** is fulfilled, it is clear that the planes of cross section must all be parallel to each other, since in this case they are moved by a pure translation. Hence the corresponding shape subtype could be termed *Translational GC*. **3** indicates that the cross section curves $\mathbf{c}(s, t)$ are related by a pure scaling function $f(s)$ relative to the origin O of the cross section plane coordinate system (i.e. the point of intersection between the cross section plane and the spine curve). The corresponding shape subtype could therefore be termed *Homogeneous GC*. **4** expresses the fact that the cross section curves are parallel curves of each other. This shape subtype we have called *Parallel Sections GC*. Finally, **5** implies that the normal $\mathbf{u}(s)$ to the cross section plane is parallel to the tangent of the spine curve at the point O of intersection between this curve and the cross section plane. Hence the corresponding shape subtype could be naturally termed *Right GC*.

Continuing downwards in Fig. 1 and combining the constraints **1**, **2** and **3** we arrive at the familiar SHGC (Straight Homogeneous GC) which is the most commonly discussed subtype of generalized cylinder in the literature (see the references above). Combining constraints **3** and **4**, it is easy to see that the only type of cross section curve that is transformed into a parallel curve by homogeneous scaling is the circle. Hence *Circular Sections GC* is the combination of “Homogeneous” and “Parallel Sections”. However, if the cross section curve is constant, this curve can be considered as scaled by unity ($f(s) = 1$ in **3**) as well as parallel displaced by zero ($Distance(s) = 0$ in **4**). Therefore we must also admit the *Constant Sections GC* as a legitimate subtype of “Homogeneous” and “Parallel sections”.

4 Twist Compensated Generalized Cylinders

The underlying geometric meaning of the constraints **1 - 5** above is intuitively clear. We now introduce a constraint that requires somewhat more of a geometric explanation. Consider the Right GC subtype defined by constraint **5**, and let the Frenet-frame of the spine-curve Γ be denoted by $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$. Since the plane of cross section in this case is perpendicular to the tangent $\mathbf{t}$ of the spine-curve, we are free to choose $\{\mathbf{i}, \mathbf{j}\} = \{\mathbf{n}, \mathbf{b}\}$ as the coordinate system for internal representation of the cross section curve C in the plane of cross section π, and $\mathbf{u} = \mathbf{t}$ as the unit normal of this plane. Consider how to express the most general orthogonal transformation $A(s)$ for the Right GC subtype. Our choice of internal coordinate system in the cross section plane π guarantees that this plane remains perpendicular to the spine-curve Γ as the plane π moves along this curve. In order to express a general orthogonal transformation $A(s)$ which is subject to this constraint, we must therefore transform the vectors $\{\mathbf{n}, \mathbf{b}\}$ by a general rotation in the cross section plane itself - i.e. by a rotation with axis $\mathbf{t}$. Denoting the angle of this "internal rotation" by $\varphi(s)$, we can express the general orthogonal transformation $A(s)$ corresponding to the Right GC subtype as :

$$A(s) = \begin{bmatrix} n_1(s) & b_1(s) & t_1(s) \\ n_2(s) & b_2(s) & t_2(s) \\ n_3(s) & b_3(s) & t_3(s) \end{bmatrix} \begin{bmatrix} \cos\varphi(s) & \sin\varphi(s) & 0 \\ -\sin\varphi(s) & \cos\varphi(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad \textbf{(12)}$$

We empasize that the representation(12) is valid - with suitably chosen internal rotation angle $\varphi(s)$ - for any surface of type Right GC. We will now restrict this type by introducing a constraint on $\varphi(s)$. We note that the the torsion τ of the spine-curve gives rise to an internal rotation of the cross section coordinate system $\{\mathbf{n}(s), \mathbf{b}(s)\}$ itself, as the cross section plane moves. From the Frenet formulas we have $\mathbf{b}'(s) = \tau\mathbf{n}(s)$. Hence the internal twist-angle $\psi(s)$ of the $\{\mathbf{n}(s), \mathbf{b}(s)\}$ coordinate system is given by the integral of the torsion of the spine-curve with respect to its arc-length s :

$$\psi(s) = \int \tau(s)\, ds \quad . \qquad \textbf{(13)}$$

Relative to some arbitrarily chosen initial position s_0, the internal twist of the cross section coordinate system can therefore be expressed as

$$\begin{bmatrix} \mathbf{n}(s) & \mathbf{b}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{n}(s_0) & \mathbf{b}(s_0) \end{bmatrix} \begin{bmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{bmatrix} . \qquad \textbf{(14)}$$

We now introduce the desired constraint on the internal rotation angle $\varphi(s)$ in (12). The subtype of Right GC that we have in mind corresponds to the following choice of $\varphi(s)$:

$$\textbf{6.} \qquad \varphi(s) = -\psi(s) = -\int \tau(s)\, ds \quad . \qquad \textbf{(15)}$$

The geometric meaning of **6** is to compensate the cross section curve C for the internal torsion (= twist) of the coordinate system $\{\mathbf{n}(s), \mathbf{b}(s)\}$ and *rotate the curve C with the same amount in the opposite direction*. Hence the corresponding subtype will be called *Twist Compensated GC*.

It turns out (see [10]) that if, in addition to **6**, the cross section curve remains constant, we generate an interesting surface type known as the *Monge surfaces*[1]. It is shown in [10] that the following two characterizations are equivalent: (i) A Monge surface is a surface that is generated by a plane (profile) curve whose plane rolls without slipping over a developable surface[2]; (ii) A Monge surface is a Constant Section Twist Compensated GC.

5 Generalized Rotational Surfaces

We will now generalize the rolling profile construction (i) for Monge surfaces to generate a subtype of GC which we call *Parallel Sections Twist Compensated GC*. Moreover, we will show why it constitutes the natural generalization of rotational surfaces that was mentioned above. We begin by choosing a family of parallel curves $\{C_p : p \in \mathbf{R}\}$ in the plane of cross section π. Then, as the latter plane rolls over the developable surface, we let the cross section curve C vary continuously among this family according to some arbitrarily chosen law $p = p(s)$, where s parametrizes the position of the cross section plane $\pi(s)$. Hence, in this case, the cross section curve is given by

$$C(s, t) = C_{p(s)}(t) \quad . \tag{16}$$

For fixed $t = t_0$, consider the curve $C(s, t_0)$, which is traced out for varying s by the point $Q(s_0, t_0)$ on the curve $C(s_0, t_0)$. Then, for a small increment ds, the corresponding displacement $dQ(s) = Q(s+ds, t_0) - Q(s, t_0)$ can be regarded as the result of two composant motions - one internal "parallel curve displacement" within the cross section plane, and one external rotating motion displacement around the corresponding generating line of the developable "contact surface" for the rolling motion of $\pi(s)$. Now, since parallel curves share the same normals, and since rotating motion of a plane traces out orthogonal trajectories of each of its points, it follows that each of these two composant motions of $Q(s, t_0)$ is perpendicular to the cross section curve $C(s, t_0)$. Therefore this orthogonality condition must hold for their resulting sum also. Hence the parametric net of (12) with the constraints of (15) and (16) will generate a family of plane parallel curves (s = const) and their orthogonal trajectories (t = const).

Finally, let us consider the parametric net of (12), (15) and (9) with the circular section constraint $g_1(t) = \cos t$, $g_2(t) = \sin t$. It is easy to see that as the spine-curve gets straightened out, this net turns into the net of parallel circles and meridians of the corresponding Straight Spine Rotational Surface. This is the reason for our claim that the CSTC and the PSTC subtypes constitute a natural two step generalization of a Straight Spine Rotational Surface. First we get rid of the straight spine and introduce a "curved axis" - while still keeping the circular perpendicular cross section curves (CSTC). Then we get rid of these circles as well, and introduce the parallel-curve perpendicular cross section curves (PSTC). In both cases the parametrization given by (12) and (15) delivers the generalized orthogonal net which corresponds to the orthogonal net of parallel circles and meridian curves in the case of a rotational possessing surface.

6 Examples and Additional Remarks

In Fig. 2 (left) we see a Circular Sections Twist Compensated GC with a cylindrical helix spine curve. This is the first step generalization of rotational surface that was mentioned above. Hence the parametric net is orthogonal but since the surface is not (in general) a Canal surface, the parametric net is not principal. The second step generalization of rotational surface is illustrated in Fig. 2 (right), which shows a surface of type Parallel Sections Twist Compensated GC (with the same circular helix spine curve). In fact, the PSTC surface on the right has been constructed from the CSTC surface on

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1. The term *Monge surface* (German: *Gesimsfläche*) should not be confused with the term *Monge patch*, which is often used in differential geometry to denote a surface coordinate patch of type $(x, y, f(x, y))$.
2. See [10] for a discussion of the fact that this surface is identical to the developable focal sheet of the Monge surface, and also the fact that possessing a developable focal sheet is a geometric characteristic of the Monge surface type.

the left by changing its circular cross section curves into an ellipse and its parallel curves. Since the two surfaces are parametrized as CSTC and PSTC , their parametric nets are both orthogonal.

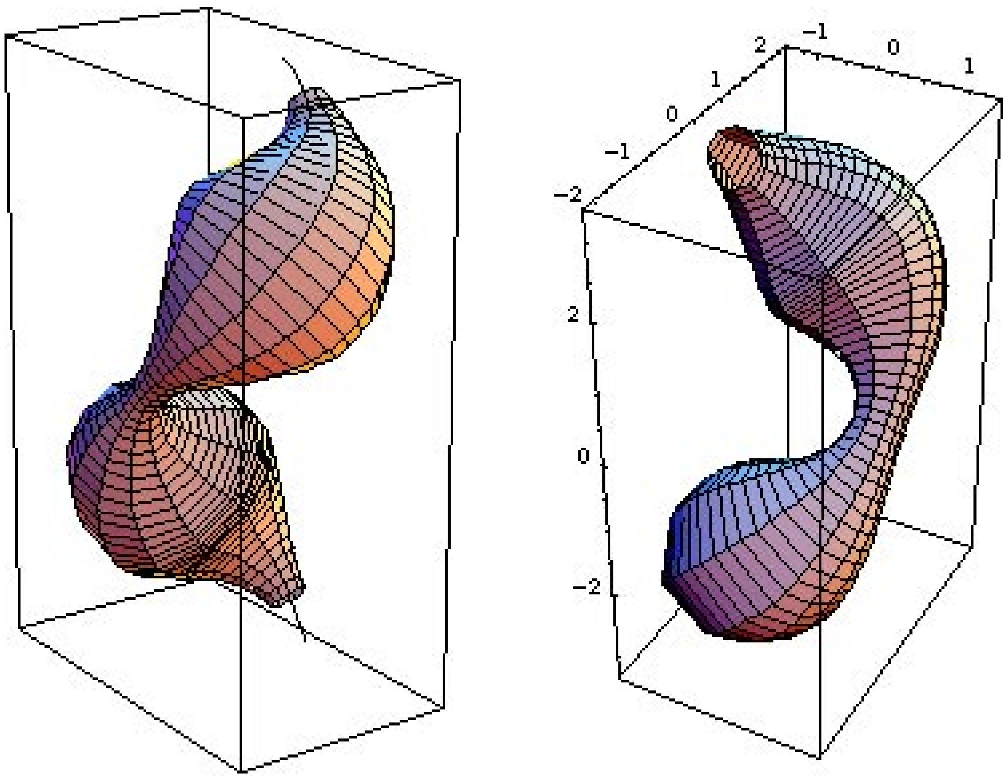

Fig. 2 GC-parametrized surface of type CSTC (left) and another one of type PSTC (right).

In Fig. 3 we present a dynamic bottom-up example of shape evolution. We start from a circular cylinder and deform it into a more general type of rotationally symmetric surface, thereby destroying the cylindrical symmetry of the original surface. This constitutes a relaxation of the parameters of the surface - the radius of the cross-section circle is no longer constant between different cross-sections. Next we destroy the rotational symmetry by deforming the straight spine-curve into a circle. We can identify the bottom-up path in the shape graph of Fig. 1 corresponding to the shape deformation that we have produced. Starting from "Circular Cylinder" we move up to "Rotational with Straight Spine" and then up to CSTC. Since the spine-curve was deformed from a straight line into a circle, the shape type of the produced surface is a combination of the types "CSTC" and "Circular Spine".

Although the question of representation uniqueness is not the topic of this paper, it is still appropriate to include a few remarks concerning this issue here. First it is important to observe the difference between the GC-representation of a surface S in terms of a spine-curve Γ and a family of cross section curves C on the one hand, and the *focal surface* (see [11]) of S on the other. Of course, the GC-parametrization of S has an inherent ambiguity, which necessitates a choice - guided by the symmetry of S - in order to obtain analytical simplicity. On the other hand, the focal surface of S is determined solely by the geometry of S and is therefore unique. Hence a GC-parametrization of S that is related to its focal surface will correspond more closely to the underlying geometry of the represented surface. [11] discusses in detail a shape-taxonomy based on the geometry of the focal surface. A family of surfaces that is sometimes mentioned in connection with generalized cylinders are the envelopes of 1-parameter families of spheres.

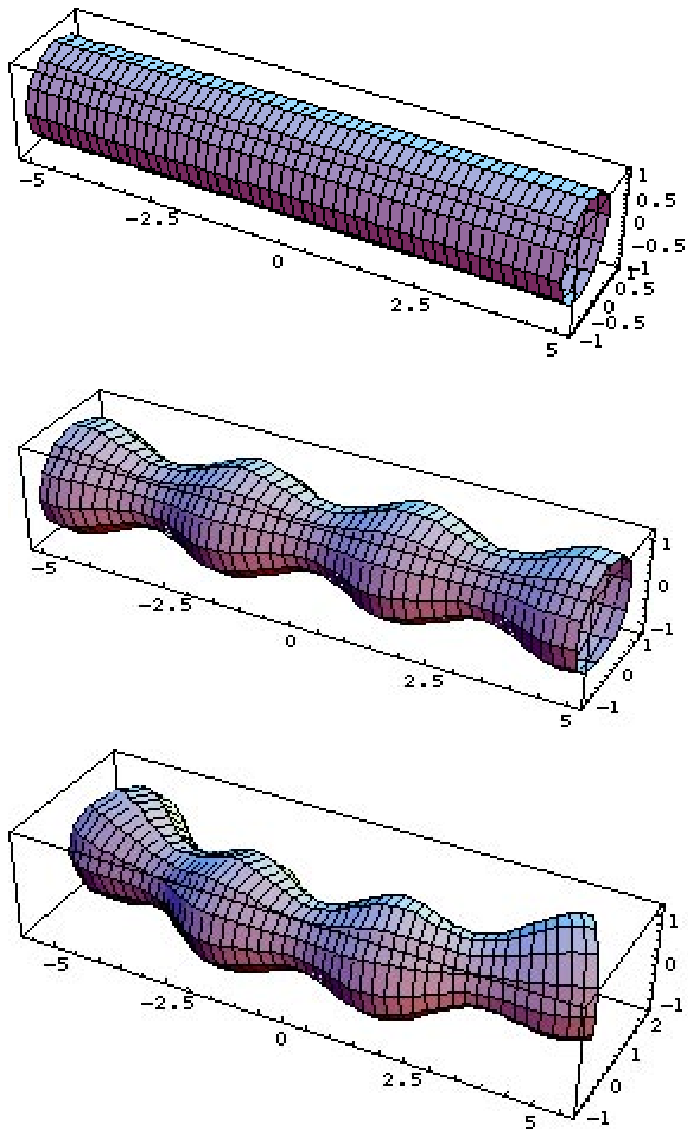

Fig. 3 A cylinder evolving its shape by breaking some of its symmetries

In the geometric literature such surfaces are often called *Canal surfaces*[1] . It might be surmised that a Canal surface should have a natural parametrization as a CSTC-type of generalized cylinder. This is however not the case, as shown in [10].

7 Summary

We have proposed an approach to systematically taxonomizing generalized cylinders by an appropriate parametrization. In this way we obtain a rich set of geometric surface classes suitable for future studies with respect to invariants and visually observable properties. By computer generated examples we have demonstrated the descriptive power of our taxonomy, which goes far beyond that of earlier

1. There is a most unfortunate divergence of terminology involved here. We follow the German tradition, which takes the term *Kanalfläche* to mean the envelope of a general 1-parameter family of spheres. When the radius is constant, the germans refer to a *Rohrenfläche*, which we have translated into *Piped surface*. Some English writers (notably Eisenhart) use the term *Canal surface* in this more restricted case of constant radius. To make matters worse, the term *Tubular surface* is also used in two ways by the geometric community ; sometimes referring to a general spherical envelope (e.g. by Eisenhart [12]), and sometimes to a spherical envelope with constant radius (e.g. by Koenderink [3]).

suggested schemes. As discussed by Naeve in [11], there are natural ways of topologizing representations of the proposed type, hence obtaining measures of similarity of evolving shapes.

8 References

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