# Geo-MAP Unification

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## 1 Introduction

The aim of this chapter is to contribute to David Hestenes' vision - formulated on his web-site[1] - of desiging a universal geometric calculus based on geometric algebra. To this purpose we introduce a framework for geometric computations which we call *geo-MAP* (*geo-Metric-Affine-Projective*) *unification*. It makes use of geometric algebra to embed the representation of euclidean, affine and projective geometry in a way that enables coherent shifts between these different perspectives. To illustrate the versatility and usefulness of this framework, it is applied to a classical problem of plane geometrical optics, namely how to compute the envelope of the rays emanating from a point source of light after they have been reflected in a smoothly curved mirror.

Moreover, in the appendix, we present a simple proof of the fact that the 'natural basis candidate' of a geometric algebra - the set of finite subsets of its formal variables - does in fact form a vector space basis for the algebra. This theorem opens the possibility of a deductive presentation of geometric algebra to a wider audience.

## 2 Historical background

Applying polynomial algebra to geometry is called algebraic geometry if the polynomials commute with each other, and geometric algebra if they don't.[2] Let us take a brief look at the historical process that has developed the design of present day relationships between geometry and algebra.[3]

1. This web-site can be located via the *Faculty Directory* of the *Depart of Physics and Astronomy*, *Arizona State University.*
2. Hence, in these mathematical field descriptions, the words 'algebra' and 'geometry' do not themselves commute.

With his work *La Geometrie*[1], published in 1637, Descartes wielded together the two subjects of algebra and geometry, which had each been limping along on its own one leg. In this process he created the two-legged subject of analytic geometry, which turned out to be capable of moving forward in big strides and even giant leaps - such as e.g. the one manifested by the differential and integral calculus of Newton and Leibnitz[2] towards the end of the 17:th century - building on work by Fermat, Wallis and Barrow[3].

But these tremendous advancements came at a considerable price. As Hestenes points out in [(20)], the careful separation between number and one-dimensional un-oriented magnitude that had been so meticulously upheld by Euclid, was thrown away by Descartes, an identification that has had fundamental consequences for the design of 'vectorial' mathematics[4]. By giving up the difference between order and magnitude, Descartes in effect created a 1-dimensional concept of direction, where only 1-dimensional geometric objects - like lines or line segments - could be oriented in a coordinate free way.

As we know, two centuries later, Grassmann discovered how to represent oriented multi-dimensional magnitudes and introduced a calculus for their manipulation[5]. Unfortunately, the fundamental nature of his contributions were not acknowledged by his contemporaries, and in the great battle over how to express multi-dimensional direction that was eventually fought between the 'ausdehnungen' of Grassmann [(15)] on the one hand and the 'quaternions' of Hamilton [(16)] on the other, the victory was carried off by the vector calculus of Gibbs [(13)] and Heaviside [(18)] - under heavy influence from the electro-magnetic theory of Maxwell [(27)]. This fact has fundamentally shaped the way we think about vectors today, and it has lead to concepts such as the familiar cross-product that we teach our mathematics students at the universities.[6]

Another example of a dominating design-pattern within present day geometrical mathematics is provided by the field of algebraic geometry. Within this field of mathematics, one describes non-commutative geometrical configurations in terms of commutative polynomial rings - using so called "varieties of zero-sets" for different sets of polynomials[7]. The urge to factor these polynomials creates an urge to algebraically close the field of coefficients, which leads to complex-valued coefficients, where one has factorization of single-variable polynomials into linear factors, and where Hilbert's so called 'nullstellensatz'[8] provides the fundamental linkage between the maximal ideals of the polynomial ring and the points of the underlying geometric space.

By adhering to this design one gains a lot of analytical power, but one also achieves two important side effects that have major consequences for the application of this kind of mathematics to geometric problems: First, one loses contact with real space,

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3. Hestenes [(20)] gives an excellent description of this process.
1. Descartes [(10)].
2. See Newton [(32), (33)] and Leibnitz [(24), (25)].
3. See [(3)] for more detail.
4. i.e. the mathematics of vectors, matrices, tensors, spinors, etc. as we know it today.
5. See Grassmann [(14)].
6. An interesting account of this development is given by Crowe in [(9)].
7. See e.g. Hartshorne: "What is algebraic geometry" [(17), pp. 55-59].
8. See e.g. Atiyah-Macdonald [(1), p. 85]

which means that algebraic geometry does not have much to say about real geometry at all, since there is no nullstellensatz to build on here. Second, since the interesting structures (= the varieties) are zero-sets of polynomials, they are mostly impossible to compute exactly, and hence must be approached by some form of iterational scheme - often using the pseudo-inverse (singular values) decomposition which is the flag-ship of computational linear algebra.

We could argue other cases in a similar way. The point is not to criticise the corresponding mathematical structures per se, but rather to underline the importance of discussing the concept of mathematical design in general - especially within the scientific communities that make use of mathematical models to represent the phenomena which they study. Bringing a powerful mathematical theory into a scientific field of study most often leads to interesting applications of that theory, but it always carries with it the risk of getting caught up in the difficulties introduced by the theory itself - rather than using it as a tool to handle the difficulties of the initial problem domain. Of course there is never a sharp distinction between these two cases, but rather a trade-off between the beneficial and the obstructional aspects of any mathematical tool.

In short, the historical process described above has resulted in a large variety of algebraic systems dealing with vectors - systems such as matrix algebra, spinor algebra, differential forms, tensor algebra, etc. etc. For many years David Hestenes has pointed out that such a multitude of computational systems for closely related conceptual entities indicate underlying weaknesses of the corresponding mathematical designs. As members of the scientific community we all share a dept of gratitude to people like David Hestenes and Garret Sobczyk, who have devoted a large part of their research to exploring various alternative designs - designs that aim to develop the legacy of Grassmann [(14), (15)] and Clifford [(4), (5), (6)] into its full algebraic and geometric potential[1]. In fact, it was the works of Hestenes [(20)] and Hestenes & Sobczyk [(21)] that brought the present authors into the field, and made us interested enough to take up active research in geometric algebra.

## 3 Geometric background

### 3.1 Affine Space

As a background to the following discussion we begin by introducing the abstract concept of an affine space followed by a concrete model of such a space to which we can apply geometric algebra. Our presentation of affine spaces follows essentially that of Snapper & Troyer [(35)], which starts out by discussing affine spaces in abstract and axiomatic mathematical terms:

**Def. 1:** The *n-dimensional affine space* over a field $K$ consists of a non-empty set $X$, an $n$-dimensional vector space $V$ over $K$, and an 'action' of the additive group of $V$ on $X$. The elements of $X$ are called *points*, the elements of $V$ are called *vectors* and the elements of $K$ are called *scalars*.

**Def. 2:** To say that the additive group of the vector space $V$ *acts* on the set $X$ means that, for every vector $v \in V$ and every point $x \in X$ there is defined a point $vx \in X$ such that

1. In this battle Hestenes and Sobczyk have been joined by a number of people. They include Rota and his co-workers Barabei and Brini [(2)], who were instrumental in reviving Grassmann's original ideas, and Sommer [(22), (36)], who plays an important role in bringing these ideas in contact with the engineering community, thus contributing to the growing number of their applications.

(i): If $v, w \in V$ and $x \in X$, then $(v + w)x = v(wx)$.
(ii): If 0 denotes the zero vector of $V$, then $0x = x$ for all $x \in X$.
(iii): For every ordered pair $(x, y)$ of points of $X$, there is one and only one vector $v \in V$ such that $vx = y$.

The unique vector $v$ with $vx = y$ is denoted by $\overrightarrow{xy}$ and we write

$$v \equiv \overrightarrow{xy} \equiv y - x \tag{1}$$

Also, it is convenient to introduce the following

**Notation:** The affine space defined by $X$, $V$, $K$ and the action of the additive group of $V$ on $X$ is denoted by $(X, V, K)$.

From now on, we will restrict $K$ to be the field of real numbers **R**. The corresponding affine space $(X, V, \mathbf{R})$ is called *real affine space*. We now introduce a model for real affine space - a model which is in fact often taken as a definition of such a space.

Let $V$ be an $n$-dimensional vector space over **R**. For the set $X$, we choose the vectors of $V$, that is, $X = V$, where $V$ is considered only as a set. The action of the additive group of $V$ on the set $V$ is defined as follows:

$$\text{If } \; v \in V \; \text{ and } \; w \in V, \text{ then } v \circ w = v + w. \tag{2}$$

It is an easy exercise to verify that

**Prop. 1:** The space $(V, V, \mathbf{R})$ as defined above, is a model for $n$-dimensional real affine space, in other words, the three conditions of Def. (2) are satisfied.

In this case one says that the vectors of $V$ act on the points of $V$ by *translation* - thereby giving rise to the affine space $(V, V, \mathbf{R})$. In linear algebra one becomes accustomed to regarding the *vector* $v$ of the vector space $V$ as an arrow, starting at the point of origin. When $V$ is regarded as an affine space, that is, $X = V$, the *point* $v$ should be regarded as the end of that arrow.

To make the distinction between a vector space and its corresponding affine space more visible, it is customary to talk of *direction vectors* when referring to elements of the vector space $V$ and *position vectors* when referring to elements of the corresponding affine space $(V, V, \mathbf{R})$.

### 3.2 Projective space

**Def. 3:** Let $V$ be an $(n+1)$-dimensional vector space. The $n$-dimensional projective space $P(V)$ is the set of all non-zero subspaces of $V$.

To each non-zero $k$-blade $B = b_1 \wedge b_2 \wedge \ldots \wedge b_k$ we can associate the linear span $\overline{B} = Linspan[b_1, b_2, \ldots, b_k]$. Hence we have the mapping from the set **B** of non-zero blades to $P(V)$ given by

$$\mathbf{B} \ni B \rightarrow \overline{B} \in P(V), \tag{3}$$

which takes non-zero $k$-blades to $k$-dimensional subspaces of $V$.

As is well known, $P(V)$ carries a natural lattice structure. Let $S$ and $T$ be two subspaces of $V$. We denote by $S \wedge T$ the subspace $S \cap T$, and by $S \vee T$ the subspace $S + T$. Moreover, let us recall the geometric algebra dual $\vee$ of the outer product $\wedge$, defined by

$$x \vee y = (\tilde{x} \wedge \tilde{y}) I = (xI^{-1}) \wedge (yI^{-1}) I \ . \quad (4)$$

We can now state the following important result:

**Prop. 2:** Let $A$ and $B$ be non-zero blades in the geometric algebra $G$. Then

$$\begin{aligned} \overline{A \wedge B} &= \bar{A} \vee \bar{B} \ , \text{ if } \ \bar{A} \wedge \bar{B} = 0 \ , \\ \overline{A \vee B} &= \bar{A} \wedge \bar{B} \ , \text{ if } \ \bar{A} \vee \bar{B} = V \ . \end{aligned} \quad (5)$$

**Proof:** See e.g. Hestenes & Sobczyk [(21)] or Svensson [(38)].

In the so called *double algebra* - also known as the *Grassmann-Cayley algebra* - the lattice structure of $P(V)$ is exploited in order to express the *join* (= sum) and *meet* (= intersection) of its subspaces[1]. In order to obtain the same computational capability within the geometric algebra $G(e_1, \ldots, e_n)$, we can introduce an alternating multilinear map called the *bracket* (or the *determinant*), given by

$$\begin{aligned} &V \times \ldots \times V \ \rightarrow \ \mathbf{R} \\ &(v_1, \ldots, v_n) \rightarrow (v_1 \wedge \ldots \wedge v_n) I^{-1} = |v_1, \ldots, v_n| \ . \end{aligned} \quad (6)$$

As an example, which we will make use of below, we have the following

**Prop. 3:** If $A, B, C, D \in G_3^1$, then

$$(A \wedge B) \vee (C \wedge D) = |A, B, C| D - |A, B, D| C \ . \quad (7)$$

**Proof:** Since both sides of (7) are multilinear in $A$, $B$, $C$, and $D$, it is enough to verify the equality for $\{A, B, C, D\} \subset \{e_1, e_2, e_3\}$. This is left as a routine exercise.

## 4 The unified geo-MAP computational framework

As is well-known, the present day vector concept is surrounded by a great deal of confusion, and we argued above that this is an indicator of its weakness of design. Symptoms range from the inability of students to discriminate between direction vectors and position vectors to heated discussions among experts as to which type of algebra that is best suited to represent vectors. Since any representational perspective has its own inherent strengths and weaknesses, it is important to be able to move between such perspectives in a consistent way, which means to remain within the same computational framework in spite of the change of representation.

In this section we demonstrate how geometric algebra provides a common background for such movement. We will explain how this background can be used to handle the interplay between euclidean (direction) vectors, affine (position) vectors and homogeneous (sub-space) vectors - such as the ones used in projective geometry.

1. See e.g. Barabei&Brini&Rota[(2)] or Svensson[(38)].

The technique for doing this we have termed *geo-Metric-Affine-Projective unification*. It is important because it allows passing from the euclidean vector space algebra into the Grassmann-Cayley algebra and then back again without changing the frame of reference. Later we will make use of the geo-MAP unification technique to compute intersections of affine sets in cartesian coordinates.

### 4.1 Geo-MAP unification

Let $V$ be a $n$-dimensional euclidean real vector space with the vectors $\{e_1, \ldots, e_n\}$ as an orthonormal basis. Denote by $G(I)$ the corresponding geometric algebra, where $I = e_1 e_2 \ldots e_n$. Moreover, let $O$ denote an arbitrary (fixed) point of the affine space $(V, V, \mathbf{R})$. We can represent $O$ by introducing a unit vector $e$ orthogonal to $\{e_1, \ldots, e_n\}$ and consider the geometric algebra $G(J)$, with unit pseudo-scalar $J = Ie$. Then it follows directly from Def. (2) that for each affine point $p \in (V, V, \mathbf{R})$ there is a unique vector $x \in V$ such that

$$p = e + x . \tag{8}$$

Moreover, the additive action of the vectors $x, y \in V$ is given by

$$(e + x) + y = e + (x + y) . \tag{9}$$

Now, by construction, we have $V = G^1(I) = \bar{I}$. Let us introduce the vector space $W$ with the corresponding relation to $J$ :

$$W = G^1(J) = \bar{J} . \tag{10}$$

Then it is clear that

$$A = e + V = \{e + v : v \in V\} \tag{11}$$

is an affine set in $W$.

We now introduce the two mappings

$$V \ni x \rightarrow {}^*x = e + x \in A , \tag{12}$$

and

$$W \setminus V \ni y \rightarrow {}_*y = (y \cdot e)^{-1} y - e \in V . \tag{13}$$

Note that $y \notin V$ ensures that $y \cdot e \neq 0$. Since the right hand side of (13) is invariant under scaling of $y$, it follows that this mapping can be extended to the 1-dimensional subspaces of $W$ (excluding the subspaces of $V$). Hence (13) induces a mapping from the affine part of projective space:

$$P(W) \setminus \{V\} \ni \bar{y} \rightarrow {}_*y \in V . \tag{14}$$

We can now make the following important observation:

$${}_*(\overline{{}^*x}) = {}_*(\overline{x + e}) = ((x + e) \cdot e)^{-1} (x + e) - e = x . \tag{15}$$

The relation (15) embodies the essence of the unified geo-MAP computational framework. It shows how to pass from a point $x$ of euclidean space - via affine space - into projective space, and then how to get to back again to the original starting point $x$. In this way the upper and lower left star operators bridge the gap betweeen the metric, affine and projective perspectives on geometry and unifies them within the same computational framework. The configuration is illustrated in Figure (1).

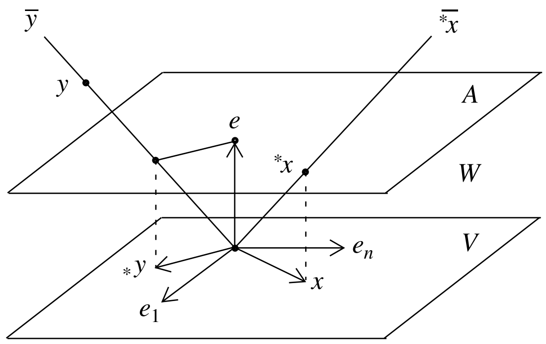


**Figure 1.** The unified geo-MAP computational framework.

Using the geo-MAP unification technique, we can start with a euclidean direction vector problem and translate it into projective space, where we can apply e.g. the techniques of the double algebra in order to find intersections of affine sets. These intersections can then be transported back to the euclidean representation and deliver e.g. cartesian coordinates of the intersection elements. In the next paragraph will apply the geo-MAP unification technique in this way.

### 4.2 A simple example

To illustrate how it works, will now apply the geo-MAP computational technique to the simple problem of finding the intersection of two lines in the affine plane.

Let $v_0, v_1, w_0, w_1 \in G^1(e_1, e_2)$ and let the two lines be determined respectively by the two point-pairs $V_i = {}^*v_i$ and $W_i = {}^*w_i$ , $i = 0, 1$ .

Making use of (7), we can express the point $p_{vw}$ of intersection between these two lines as:

$$p_{vw} = {}_*((V_0 \wedge V_1) \vee (W_0 \wedge W_1)) = {}_*(|V_0, V_1, W_0| W_1 - |V_0, V_1, W_1| W_0) \quad \textbf{(16)}$$

and the two brackets that appear in (16) can be written as:

$$\begin{aligned} |V_0, V_1, W_0| &= |v_0 + e, v_1 + e, w_0 + e| \\ &= |v_0 - w_0, v_1 - w_0, w_0 + e| \\ &= |v_0 - w_0, v_1 - w_0| \\ &= ((v_0 - w_0) \wedge (v_1 - w_0))\, e_2 e_1 \end{aligned} \quad \textbf{(17)}$$

and analogously

$$|V_0, V_1, W_1| = ((v_0 - w_1) \wedge (v_1 - w_1))\, e_2 e_1 \, . \quad \textbf{(18)}$$

Writing $\alpha = |V_0, V_1, W_0|$ and $\beta = |V_0, V_1, W_1|$, (16) takes the form:

$$\begin{aligned} p_{vw} &= {}_*(\alpha W_1 - \beta W_0) \\ &= ((\alpha W_1 - \beta W_0) \cdot e)^{-1} (\alpha W_1 - \beta W_0) - e \\ &= (\alpha - \beta)^{-1} (\alpha w_1 + \alpha e - \beta w_0 - \beta e) - e \\ &= \frac{\alpha w_1 - \beta w_0}{\alpha - \beta} . \end{aligned} \tag{19}$$

Here we have a good example of the geo-MAP unification technique at work. Taking the upper star of the euclidean direction vectors $w_0$ and $w_1$, they are brought into the affine part of projective space. Here they can be subjected to the double algebraic lattice operations of $\wedge$ and $\vee$, in this particular case in the combination expressed by (7).

### 4.3 Expressing euclidean operations in the surrounding geometric algebra

We will end this section by showing how to embed some important euclidean direction vector operations within the surrounding geometric algebra $G_4$ - namely the difference and the cross-product operations. In this way the ordinary euclidean algebra of (cartesian) direction vectors can be emulated in geometric algebra.

Consider the euclidean direction vector $x \in G_3^1 \subset G_4^1$ with its cartesian coordinate expansion given by $x = x_1 e_1 + x_2 e_2 + x_3 e_3$. Recall that $I = e_1 e_2 e_3$ and $J = Ie$. By (14), the corresponding affine position vector $X \in (V, V, \mathbf{R}) \subset G_4^1$ is expressible as $X = x + e$.

With these definitions of $x$ and $X$, and the corresponding definitions for $y$ and $Y$, we will now deduce two formulas that connect the euclidean direction vector algebra of Gibbs with the surrounding geometric algebra $G_4$:

**Prop. 4:** The euclidean cross product vector can be expressed in $G_4$ as:

$$x \times y = (e \wedge X \wedge Y) J . \tag{20}$$

**Proof:** From the definition of the cross product, it follows directly that

$$x \times y = (x \wedge y) I^{-1} . \tag{21}$$

Plugging $X = x + e$ and $Y = y + e$ into the right-hand-side of (20) now gives

$$\begin{aligned} (e \wedge X \wedge Y) J &= (e \wedge (x + e) \wedge (y + e)) J = (e \wedge x \wedge y) J \\ &= e (x \wedge y) J = (x \wedge y) eIe = (x \wedge y) I^{-1} \\ &= x \times y . \end{aligned} \tag{22}$$

**Prop. 5:** The euclidean difference vector can be expressed in $G_4$ as:

$$y - x = e \cdot (X \wedge Y) . \tag{23}$$

**Proof:** Expanding again the right-hand-side of (23), we can write

$$\begin{aligned} e \cdot (X \wedge Y) &= e \cdot ((x+e) \wedge (y+e)) = e \cdot (x \wedge y + xe + ey) \\ &= e \cdot (xe + ey) = e \cdot (ey - ex) = \{ey \text{ and } ex \in G_2\} \\ &= \langle eey \rangle_1 - \langle eex \rangle_1 = y - x . \end{aligned}$$

Formulas such as (20) and (23) are useful for translating a geometric problem from one representation into another. Moreover, since for a 1-vector $v$ and a blade $B$ we have $v \cdot B + v \wedge B = vB$, it follows that (20) and (23) can be combined into:

$$e(X \wedge Y) = y - x + (x \times y) J^{-1} \tag{24}$$

or

$$X \wedge Y = e(y - x) + (x \times y) I . \tag{25}$$

The expression (25) indicates interesting relationships between ordinary direction vector algebra and various forms of generalized complex numbers. However, to pursue this topic further is outside the scope of the current text.

# 5 Applying the geo-MAP technique to geometrical optics

In order to illustrate the workings of the unified geo-MAP computational technique, we will now apply it to a classical problem of geometrical optics. It was first treated by Tschirnhausen and is known as the problem of *Tschirnhausen's caustics*.

## 5.1 Some geometric-optical background

Since light is considered to emanate from each point of a visible object, it is natural to study optics in terms of collections of point sources. In geometrical optics, a point source is considered as a set of (light) rays - i.e. a set of directed half-lines - through a point. But a point source does not (in general) retain its 'pointness' as it travels through an optical system of mirrors and lenses. When a point source of in-coming light is reflected by a mirror or refracted by a lens, the out-going rays will in general not pass through a point. Instead, they will be tangent to two different surface patches, together called the *focal surface* of the rays[1].

The importance of focal surfaces in geometrical optics is tied up with a famous theorem due to Malus and Dupin[2]. In order to understand what this theorem says, we introduce a geometric property that is sometimes possessed by a set of lines:

**Def. 4:** A two-parameter family of curves $K$ is said to form a *normal congruence*, if there exists a one-parameter family of smooth surfaces $\Omega$ such that each surface of the family $\Omega$ is orthogonal to each curve of the family $K$.

A surface in $\Omega$ is called an *orthogonal trajectory* to the curves of the family $K$. A point field $K_P$ is of course an example of a *normal congruence of lines*, the orthogonal trajectories being the one-parameter family of concentric spheres $\Omega_P$ with centre $P$. Furthermore, the rays of $K_P$ carry a direction, which varies continuously when we pass from one ray to its neighbours. Such a family of lines is called a *directed normal congruence*.

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1. For a survey of the theory of focal surfaces, we refer the reader to Naeve [(31)].
2. see e.g. Lie [(26)].

Now, the theorem of Malus and Dupin can be formulated as follows:

**Prop. 6:** A directed normal congruence of lines remains directed and normal when passing through an arbitrary optical system.

In optics, the orthogonal trajectories of the light rays are called *wave fronts*, and the Malus-Dupin theorem can be expressed by stating that *any optical system preserves the existence of wave fronts*.

### 5.2 Determining the second order law of reflection for planar light rays

In what follows below we will restrict to the plane and consider one-parameter families of rays that emanate from a planar point source and then are reflected by a curved mirror in the same plane. We will deduce an expression that connects the curvatures of the in-coming and out-going wave fronts with the curvature of the mirror at the point of impact. Since curvature is a second-order phenomenon, it is natural to call this expression the second order law of reflection - as opposed to the first order law, that expresses only the direction of an outgoing ray as a function of the direction of the incoming ray and the direction of the mirror normal.

Let us begin by recalling some classical concepts from the differential geometry of plane curves.

**Def. 5:** Consider a one-parameter family of smooth curves $F(c)$ in the same plane (with $c$ as the parameter). If there is a curve $\Gamma$ which has the property of being tangent to every curve of the family $F(c)$ in such a way that each point $\Gamma(t_0)$ is a point of tangency of exactly one curve $F(c_0)$, then the curve $\Gamma$ is called the *envelope* of the family of curves $F(c)$.

**Def. 6:** Consider a smooth plane curve $M$. At each point $m(s)$ of this curve there is an osculating circle with centre point $r(s)$, called the centre of curvature of the point $m(s)$. When we vary $s$ (i.e. when we move along the curve $M$) the point $r(s)$ will describe a curve $E$ called the *evolute* of the original curve $M$. Reciprocally, the curve $M$ is called the *evolvent* (or the *involute*) of the curve $E$.

In 2-dimensional geometrical optics, a point source of light corresponds to a *pencil of rays.* After having been reflected or refracted by various mirrors and lenses, these rays will in general be tangent to a curve, called a *caustic curve* in optics[1].This is the kind of bright, shining curve that we can observe when sunlight is reflected e.g. in a cup of tea.

**Def. 7:** A certain angular sector of a plane pencil of light-rays is made to be incident on a smoothly curved mirror $M$ in the same plane. After being reflected by $M$, the light-rays of this sector are all tangent to the caustic curve forming their envelope. Such a 1-parameter family of light-rays will be referred to as a *tangential sector* of rays.

**Note:** In view of the discussion above, we can conclude that the caustic curve of a tangential sector of rays is at the same time the *envelope of the rays* and the *evolute of their orthogonal trajectories*.

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1. See e.g. Cornbleet [(7)] or Hecht&Zajac [(19)].

Let us consider a tangential sector of rays $L_{12} = \{l(s) : s_1 < s < s_2\}$ with caustic curve $C_{in}$ whose rays are incident on a smoothly curved mirror $M$ between the points $m(s_1)$ and $m(s_2)$ as depicted in Figure (2).

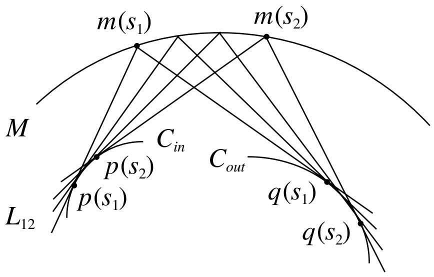


**Figure 2.** The tangential sector $L_{12}$ is reflected by the mirror $M$.

Two closely related rays $l(s)$ and $l(s+ds)$ will intersect in a point that is close to the caustic curve $C_{in}$ and when $l(s+ds)$ is brought to coincide with $l(s)$ by letting $ds \to 0$ their point of intersection will in the limit fall on the caustic point $p(s)$. Hence we can regard the caustic curve passing through $p$ as the locus of intersection of neighbouring rays, where the term 'neighbouring' refers to the limiting process just described[1].

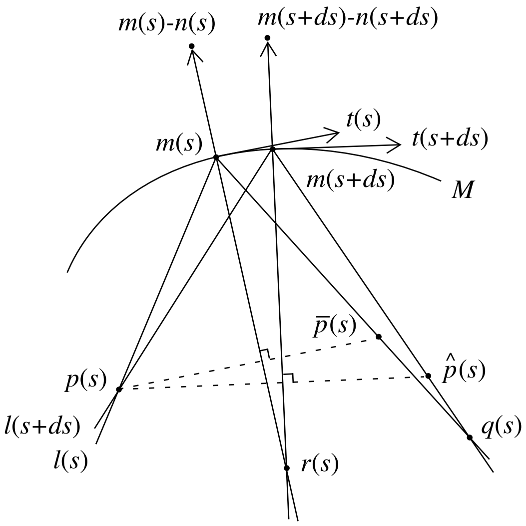


**Figure 3.** Two neighboring rays $l(s)$ and $l(s+ds)$ intersecting at $p(s)$ and their respective reflections intersecting at $q(s)$.

In Figure (3) the symbols $t$ (=tangent) and $n$ (=normal) denote euclidean direction vectors, and $m$, $p$, $q$, $r$ denote affine points (= position vectors). The symbol $s$ denotes the parameter that connects corresponding points and lines. The two rays $l(s)$ and

1. This is an example of an intuitive (and very useful) way to think about geometrical entities that has been used by geometers ever since the days of Archimedes. Unfortunately it has no formal foundation in classical analysis. Since infinitesimal entities (like $dx$ and $dy$) do not exist in the world of real numbers, a line in a continuous, (real), one-parameter family can have no neighbour. However, the concept of neighbouring geometrical objects can be made rigorous by the use of so called non-standard analysis.

$l(s+ds)$ can be thought of as forming an infinitesimal sector with its vertex $p$ on the caustic curve of $l$. Within this sector, the corresponding (infinitesimal) parts of the wave-fronts are circular and concentric around $p$. The point $p$ can therefore be regarded as the focal point of this infinitesimal sector, i.e. the *local focal point* of the wave fronts in the direction given by $l(s)$.

Having thus established some terminology and notation, we now turn to Tschirnhausen's problem, which is concerned with determining the point $q(s)$ on the reflected caustic $C_{out}$ that corresponds to the point $p(s)$ on the in-caustic $C_{in}$. It can be solved by making use of the theory of envelopes[1], but here we will give a more intuitive and straight-forward solution that makes use of the unified geo-MAP technique - in combination with ordinary Taylor expansion - to compute a point of the reflected caustic $C_{out}$ as the intersection of two neighbouring reflected rays.

From Sec. (4.2) we recall the expression (19) for the point of intersection of the two lines determined by the two pairs of points $V_i = {}^*v_i$ and $W_i = {}^*w_i$, $i = 0, 1$, where the determinants $\alpha$ and $\beta$ appearing in (19) are given by (17) and (18).

For the sake of convenience (with respect to the computations that follow) let us fix the coordinate system so that

$$m(s) = O\,,\ t(s) = e_1\,, n(s) = -e_2\,. \tag{26}$$

**Note:** Since the point $m$ is to be considered as the point $O$ of origin for the space of direction vectors, we will write $p$ and $q$ in order to denote the direction vectors $p - m$ and $q - m$ respectively. This will shorten the presentation of the computations considerably. When we have finished, we will restore the correct direction vector expressions and present the desired result in a way that distinguishes clearly between position vectors and direction vectors.

Following these preliminaries, we will now make use of (19) to compute the point of intersection $q(s)$ of two neighbouring reflected rays.

From classical differential geometry[2] we recall the so called Frenét equations for a curve $M$ :

$$\begin{aligned} \dot{t}(s) &= \frac{1}{\rho} n(s) \\ \dot{n}(s) &= -\frac{1}{\rho} t(s)\ . \end{aligned} \tag{27}$$

Here $t(s)$ and $n(s)$ are the unit tangent respectively the unit normal to $M$ at the point $m(s)$, and $\rho = \rho(s) = |r(s) - m(s)|$ is the radius of curvature of $M$ at this point.

Moreover, since $s$ denotes arc-length on $M$, we have $\dot{m}(s) = t(s) = e_1(s)$, and (27) can be written in the form:

1. see Kowalewski[(23), pp. 50-54].
2. See e.g. Eisenhart [(11)] or Struik [(37)].

$$\dot{e}_1(s) = -\frac{1}{\rho}e_2(s)$$
$$\dot{e}_2(s) = \frac{1}{\rho}e_1(s) \quad . \tag{28}$$

The reflected rays corresponding to the parameter values $s$ and $s + ds$ are determined by the two point-pairs $\{m(s), \bar{p}(s)\}$ respectively $\{m(s+ds), \hat{p}(s)\}$, where the points $\bar{p}(s)$ and $\hat{p}(s)$ are constructed by reflecting the point $p(s)$ in the tangent $t(s)$ at the point $m(s)$ respectively the tangent $t(s+ds)$ at the point $m(s+ds)$. Using the reflection formula for vectors[1], we can write

$$\bar{p}(s) - m(s) = -e_1(s)(p(s) - m(s))e_1(s)^{-1}$$
$$\hat{p}(s) - m(s+ds) = -e_1(s+ds)(p(s) - m(s+ds))e_1(s+ds)^{-1} \quad . \tag{29}$$

Suppressing the dependence of $s$ and recalling from (26) that $m(s) = 0$, (29) takes the form

$$\bar{p} = -e_1 p e_1$$
$$\hat{p} - \dot{m}ds = -(e_1 + \dot{e}_1 ds)(p - \dot{m}ds)(e_1 + \dot{e}_1 ds) + O(ds^2) \tag{30}$$

where $O(ds^2)$ denotes the well-known ‘big-oh’ ordo-class of functions $f$, that is

$$f \in O(ds^2) \Leftrightarrow |f(s+ds) - f(s)| \le K|ds|^2 \tag{31}$$

for some constant $K = K(s)$. Expanding the right hand side of the second equation of (30) gives

$$\hat{p} - \dot{m}ds = -(e_1 p e_1 + (e_1 p \dot{e}_1 + \dot{e}_1 p e_1 - e_1 \dot{m} e_1)ds) + O(ds^2) \quad . \tag{32}$$

Now $\dot{e}_1 p e_1 = e_1 p \dot{e}_1$, and since $\dot{m} = e_1$, we have $e_1 \dot{m} e_1 = e_1$. Therefore we can write

$$\hat{p} = -e_1 p e_1 + (e_1 + e_1 - e_1 p \dot{e}_1 - \dot{e}_1 p e_1)ds + O(ds^2)$$
$$= -e_1 p e_1 + 2(e_1 - e_1 p \dot{e}_1)ds + O(ds^2) \quad . \tag{33}$$

In order to make use of the intersection formula (19), we first compute

$$\alpha = (v_0 - w_0) \wedge (v_1 - w_0)e_2 e_1 = (-\dot{m}ds) \wedge (\bar{p} - \dot{m}ds)e_2 e_1 + O(ds^2)$$
$$= -((\dot{m}ds) \wedge \bar{p})e_2 e_1 + O(ds^2) = (\bar{p} \wedge \dot{m})e_2 e_1 ds + O(ds^2) \tag{34}$$

and

$$\beta = (v_0 - w_1) \wedge (v_1 - w_1)e_2 e_1 = (0 - \hat{p}) \wedge (\bar{p} - \hat{p})e_2 e_1$$
$$= -(\hat{p} \wedge \bar{p})e_2 e_1 = (\bar{p} \wedge \hat{p})e_2 e_1 \quad . \tag{35}$$

Moreover, if we split the vector $p$ into the components $p = \psi_1 e_1 + \psi_2 e_2$ and make use of the fact that $\dot{m} = e_1$, we get from (34):

1. See Hestenes [(20), p. 278]

$$\alpha = -\psi_2 ds + O(ds^2) \ , \tag{36}$$

and a rather lengthy but straightforward calculation gives

$$\bar{p} \wedge \hat{p} = -2\,(|\dot{e}_1|p^2 + \psi_2)\, e_1 e_2 ds + O(ds^2) \ . \tag{37}$$

Plugging (37) into (35) gives

$$\beta = -2\,(|\dot{e}_1|p^2 + \psi_2)\, ds + O(ds^2) \ . \tag{38}$$

Finally, by making use of the intersection formula (19), we arrive at the following expression for the local focal point $q = q(s)$ of the reflected wave front corresponding to the local focal point $p = p(s)$ of the incident wave front:

$$\begin{aligned} q &= \frac{1}{\alpha - \beta}(\alpha w_1 - \beta w_0) = \frac{1}{\alpha - \beta}(\alpha \hat{p} - \beta m(s + ds)) \\ &= \frac{(\psi_1 e_1 - \psi_2 e_2)}{1 + 2|\dot{e}_1|\dfrac{p^2}{\psi_2}} + O(ds) \ . \end{aligned} \tag{39}$$

In order to restore this result - as we promised above - to a logically consistent and coordinate-free form, we must now substitute $p - m$ for $p$ and $q - m$ for $q$ in (39). Performing this substitution, we get

$$q - m = \frac{(\psi_1 e_1 - \psi_2 e_2)}{1 + 2|\dot{e}_1|\dfrac{(p-m)^2}{\psi_2}} \ . \tag{40}$$

Observe that if $|\dot{e}_1| \to 0$ , i.e. if the mirror becomes plane, (40) reduces to the familiar law of planar reflection:

$$q_t - m = \psi_1 e_1 - \psi_2 e_2 \, , \tag{41}$$

where the point $q_t$ is the reflection of $p$ in the straight line mirror $t$ that is tangent to the mirror $M$ at the point $m$. Hence, recalling that $\psi_i = (p - m) \cdot e_i$ and that $|\dot{e}_1| = |\ddot{m}| = 1/\rho$ , we can express the relationship between the corresponding points $p \in C_{in}$ and $q \in C_{out}$ in the following way:

$$\begin{aligned} q - m &= \frac{((p-m)\cdot e_1)\, e_1 - ((p-m)\cdot e_2)\, e_2}{1 + 2|\ddot{m}|\dfrac{(p-m)^2}{(p-m)\cdot e_2}} \\ &= \frac{((p-m)\cdot t)\, t - ((p-m)\cdot n)\, n}{1 - 2|\ddot{m}|\dfrac{(p-m)^2}{(p-m)\cdot n}} \end{aligned} \tag{42}$$

This relation expresses the second order law of reflection for plane geometrical optics. We summarize Tschirnhausens result in the following

**Prop. 7:** Let $C_{in}$ be the caustic curve of a plane tangential sector of rays that is incident on a plane-curve mirror *M* (located in the same plane) in such

a way that the ray which touches $C_{in}$ at the point $p$ is intercepted by $M$ in the point $m$, where the unit-tangent, unit-normal Frenét frame for the curve $M$ is given by the vectors $t$ and $n$ (according to an arbitrarily chosen incremental parametric direction of $M$).
Under these conditions, the point $q$ which corresponds to $p$, that is the point $q$ where the reflected ray from $m$ touches the caustic curve $C_{out}$, is given by the expression

$$q - m = \frac{((p-m)\cdot t)\,t - ((p-m)\cdot n)\,n}{1 - 2|\dot{m}|\dfrac{(p-m)^2}{(p-m)\cdot n}} \quad . \tag{43}$$

### 5.3 Interpreting the second order law of reflection geometrically

In order to illustrate the geometric significance of the second order reflection law given by (43), we will interpret it in projective geometric terms. In Figure (4), $p$, $\bar{p}$, $m$ and $q$ have the same meaning as before, and $\bar{r}$ denotes the result of projecting the point $r$ orthogonally onto the reflected ray through the point $m$ with direction $\overrightarrow{mq} = q - m$.

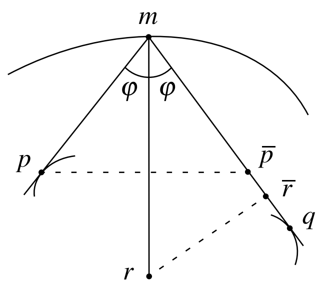


**Figure 4.** Overview of the Tschirnhausen configuration.

Introducing the angle $\varphi$ between the incident ray of direction $\overrightarrow{pm} = m - p$ and the corresponding mirror normal $e_2$, we note the following relations between the participating magnitudes:

$$\begin{aligned} q_t - m &= m - \bar{p} \\ (p-m)\cdot e_2 &= -|p-m|\cos\varphi \\ |\bar{r} - m| &= |r - m|\cos\varphi = \rho\cos\varphi \\ |\bar{p} - m| &= |p - m| \ . \end{aligned} \tag{44}$$

Taking (44) into accout, the reflection law (43) can be expressed as

$$q - m = \frac{\bar{p} - m}{\dfrac{2|p-m|}{\rho\cos\varphi} - 1} \tag{45}$$

and taking the modulus of both sides of (45), we can write

$$\frac{1}{|\bar{p}-m|} \pm \frac{1}{|q-m|} = \frac{2}{\rho\cos\varphi} = \frac{2}{|\bar{r}-m|}. \quad (46)$$

The sign in the left hand side of (46) corresponds to the sign of the denominator in the right hand side of (45).

Since, by (26), our coordinate system $\{e_1, e_2\}$ has its point of origin at $m$, the formula (46) expresses the fact that the points $\bar{p}$ and $q$ separate the points $m$ and $\bar{r}$ harmonically, that is, these two pairs of points constitute a harmonic 4-tuple. This is the form in which Tschirnhausen presented his reflection law[1].

# 6 Summary and conclusions

## 6.1 The geo-MAP unification technique

In Sec. (4.1) we introduced the unified geo-MAP computational framework - inspired by classical projective line geometry.[2] We then demonstrated how the geo-MAP framework provides a way to represent the metric (= euclidean), affine and projective aspects of geometry within the same geometric algebra, and how this representation creates a computational background for performing coherent shifts between these three different geometrical systems.

In (12) we showed how to pass from a euclidean point (= direction vector), to the corresponding affine part of projective space, and in (13) we showed how to get back again from the finite part of projective space to the original euclidean point that we started with. The proof that this works was provided by (15). Formulas (12) and (13) are key formulas underlying many of our later computations. Because of their great practical utility in combining the powers of the ordinary euclidean direction vector algebra with those of the Grassmann-Cayley algebra, we feel that they should be of particular interest to the engineering community.

In Sec. (4.3) we showed how to embed the basic (euclidean) direction vector algebra into the surrounding geometric algebra. The formulas (20) and (23) - combined in (24) or (25) - illustrate the interplay between the euclidean operations of vector addition and Gibbs' cross-product on the one hand - and the operations of geometric, outer and inner product on the other. Such formulas as these we have not seen anywhere else.

As an illustrative application of the unified geo-MAP computational technique, we applied it in Sec. (5) to a classical problem of plane geometrical optics called 'Tschirnhausens problem', which is concerned with determining the envelope of the rays from a point source of light after their reflection in a smoothly curved mirror.

Using the geo-MAP technique in combination with ordinary Taylor expansion, we computed the desired envelope as the locus of intersection of 'neighboring' rays, i.e. rays that differ infinitesimally from one another. In this way we deduced the expression (43), which could be termed the "second order law of reflection", since it expresses the curvature relations between the in-coming and out-going wave fronts and the curved mirror.

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1. See Kowalewski [(23). p. 51].
2. see e.g. Sauer [(34)], Naeve [(28)] or Naeve & Eklundh [(30)].

Although, in the planar case, the same result can be achieved using envelopes, the geo-MAP framework has the advantage of being applicable in higher dimensions. For example, in 3 dimensions, it can be used in order to perform the corresponding computations - relating the points on the respective focal surfaces of an in-coming and out-going normal congruence of rays to the corresponding points on the focal surface of the normals to the mirror. However, the complexity of such computations have made it necessary to exclude them here.

### 6.2 Algebraic & combinatorial construction of a geometric algebra.

As a didactic comment on how to teach geometric algebra, we present - in the appendix - a constructional proof of the fact that the 'expected basis elements' of a geometric algebra $G$ -i.e. the set of finite subsets of its formal variables - actually do form a basis for $G$. This is done in Prop. (8) and Prop. (9), leading up to Def. (10), where we define a geometric algebra by constructing it.

Our construction enables the possibility of a logically self-contained description of geometric algebra which does not require such high levels of abstraction as in the traditional tensor algebra approach, and which should therefore be accessible to a wider audience. In our opinion, the main reason for the lack of such a presentation in the present literature is the difficulties encountered in establishing a vector space basis for a geometric algebra.

Using this approach to presenting geometric algebra, we do not have to worry about the question of whether there exists any algebraic structure that is capable of implementing the desired specifications. We are therefore free to take the 'basis approach', both to defining different operations on the algebra as well as to proving their structural properties. In our opinion this greatly simplifies a deductive presentation of geometric algebra to students.

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## 8 Appendix: Construction of a geometric algebra

Let $R$ be a commutative ring, and let $\{E, <\}$ be a totally ordered set. The non-commutative ring of polynomials over $R$ in the formal variables $E$ is denoted by $R\{E\}$, and the set of monomials and the set of terms in the ring $R\{E\}$ is denoted by $M$ respectively by $T$.

Moreover, let $S = \{-1, 0, 1\} \subset R$, and let $\operatorname{sgn} : T \to R$ be a mapping from $T$ to $R$ with values in $S$, i.e. $\operatorname{sgn}(t) \in S \subset R$.

**Def. 8:** We say that the pair $(e, e') \in E^2$ is an *involution* in the term $t$, if there exist terms $t', t'', t'''$ with $t = t'et''e't'''$ such that either $e' < e$ or $e' = e$ , $\operatorname{sgn}(e) = -1$

**Notation:** The number of inversions in the term $t$ is denoted by $\operatorname{inv}(t)$.

We now define a mapping $\mu : T \to R$ in the following way:

- $\mu(rm) = r\mu(m)$ , where $r \in R$ , $m \in M$ .
- $\mu(m) = 0$ , if *m* contains at least two occurrences of some $e \in E$ with sgn($e$)=0.
- $\mu(m) = (-1)^{\mathrm{inv}(m)}$ otherwise.

We also introduce a reduction rule →, i.e. a binary relation on *T* by making the following

**Def. 9:** $t \to t'$ , where $t, t' \in T$ , if there exist terms $t_1, t_2 \in T$ and $e_1, e_2 \in E$ such that $t = t_1 e_2 e_1 t_2$ , $e_1 \le e_2$ , and where
$t' = -t_1 e_1 e_2 t_2$ , if $e_1 < e_2$ , or
$t' = \mathrm{sgn}(e_1) t_1 t_2$ , if $e_1 = e_2$ .

**Notation:** If no $t'$ exists in *T* such that $t \to t'$ , we write $t|$ , and

if $t \to t_1 \to \ldots \to t_k$ we write $t \overset{*}{\to} t_k$ .

By inspection, we observe that

$$\mathrm{inv}(m_1 e e m_2) = \mathrm{inv}(m_1 m_2) + \mathrm{inv}(ee) + 2N \text{ , for some } N \in \mathbf{N} , \tag{47}$$

and that

$$\mathrm{inv}(m_1 e_2 e_1 m_2) = \mathrm{inv}(m_1 e_1 e_2 m_2) + 1 \text{ , if } e_1 < e_2 . \tag{48}$$

From (47) and (48) we can conclude that if $t \to t'$ , we have

$$\mu(t) = \mu(t') . \tag{49}$$

We can now state the following

**Prop. 8:** For each *t* in *T* there exists a unique $t' = \mathrm{red}(t)$ in *T*, such that $t \to t'|$ .

**Proof:** We start by proving uniqueness. If $\mu(t) = 0$ then obviously $\mu(t') = 0$ .

Let $t = rm \overset{*}{\to} r'm'|$ , where $\mu(m) \ne 0$ . Then, by inspection, $m' = e_1 e_2 \ldots e_n$ , where $e_1 < e_2 < \ldots < e_n$ , and $\{e_1, \ldots, e_n\}$ is the set of *e*:s in *E* occurring an *odd* number of times in *m*. If this set is empty, we put $m' = 1$ . Hence, $m'$ is unique. Moreover, $\mu(t) = r\mu(m) = r'\mu(m')$ which shows that $r'$ is unique. This finishes the uniqueness part of the proof.

For the proof of the existence part, we observe that if $t_1 \to t_2$ , then we have

$$\deg(t_1) + \mathrm{inv}(t_1) > \deg(t_2) + \mathrm{inv}(t_2) \tag{50}$$

Hence every reduction chain $t \overset{*}{\to} t_k$ is finite, which proves the existence of $t'$ . <<>>

From Prop. (8) we can directly conclude:

**Prop. 9:** Let $B$ denote the set of monomials $m$ in $R\{E\}$ such that $m|$ . Then $B$ is in one-to-one correspondence with the set of finite subsets of $E$.

**Notation:** Let $B_n$ denote the set of monomials in $B$ of degree $n$. The $R$-modules generated by $B$ and $B_n$ are denoted by $G$ and $G_n$ respectively.

We now turn $G$ into a ring by introducing an $R$-bilinear mapping (multiplication)

$$\begin{aligned} &G \times G \to G \\ &(x, y) \to x \circ y \end{aligned} \tag{51}$$

in the following way:

By $R$-bilinearity, it is enough to define $m_1 \circ m_2$ for $m_1, m_2 \in B$ . We do so by defining

$$m_1 \circ m_2 = \operatorname{red}(m_1 m_2) \ . \tag{52}$$

We then have

$$m_1 m_2 m_3 \xrightarrow{*} m_1 (m_2 \circ m_3) \xrightarrow{*} m_1 \circ (m_2 \circ m_3) = \operatorname{red}(m_1 m_2 m_3) \,\Big| \ , \tag{53}$$

and

$$m_1 m_2 m_3 \xrightarrow{*} (m_1 \circ m_2)\, m_3 \xrightarrow{*} (m_1 \circ m_2) \circ m_3 = \operatorname{red}(m_1 m_2 m_3) \,\Big| \ . \tag{54}$$

Since $\operatorname{red}(m_1 m_2 m_3)$ is unique, it follows that the product $\circ$ is associative.

**Def. 10:** The ring $(G, \circ)$ is called a *geometric algebra* (or a *Clifford algebra*).

**Notation:** The product $\circ$ is called the *geometric product*, and it is usually written as a concatenation. Following this convention, we will from now on write $xy$ for the product $x \circ y$, i.e.

$$xy \equiv x \circ y \tag{55}$$

We can reformulate Prop. (9) as

**Prop. 10:** Let $G$ be a geometric algebra over $R$ with formal variables $E$. Then $G$ has an $R$-module basis consisting of the set of all finite subsets of $E$.

Moreover, it can be shown that the following holds:

**Prop. 11:** Let $E'$ be another set, totally ordered by $<'$, and let the mapping $\operatorname{sgn}' : E' \to R$ satisfy the condition $\operatorname{card}(E^s) = \operatorname{card}(E'^s)$ , where $E^s = \{e \in E : \operatorname{sgn}(e) = s\}$ and $E'^s = \{e' \in E' : \operatorname{sgn}'(e') = s\}$ . Then $G$ and $G'$ are isomorphic as geometric algebras.

One way to establish this isomorphism is to show that if $J$ is the ideal generated by $\{e^2 - \operatorname{sgn}(e) , ee' + e'e \ : \ e, e' \in E \ , \ e \neq e'\}$ , then $G$ is isomorphic to $R\{E\}/J$ .

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