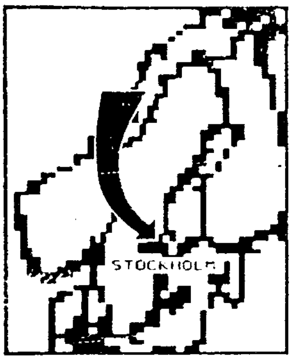

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Estimating the N-Dimensional Motion of a $(N - 1)$-Dimensional Hyperplane from Two Matched Images of $(N + 1)$ of Its Points

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Introduction

Motion constitutes one of the most important cues in our visual perception of the surrounding world. Hence, understanding motion is of great interest also in machine vision. Image motion may be due to object motion or viewer motion. Deriving object motion from image sequences is therefore a crucial problem in computational vision and a substantial amount of work has been devoted to this subject, see e.g. [2] — [9].

In this paper we study the problem of estimating the motion of a plane or a planar patch. Important contributions to this topic have been made by Tsai, Huang and Zhu who in [4], [5] and [6] concern themselves with determining the 3D-motion parameters of a rigid planar patch from two matched images of some of its points. In [5], using trigonometric representation of 3D orthogonal matrices and carrying out a number of ingeneous calculations, they arrive at explicit formulae for the motion parameters, given eight so called *pure parameters* of the motion. And in [6] they show that the observation of four points is enough to determine these eight pure parameters.

Although this work is solid and well carried out, we feel that it suffers from some serious limitations. First, the formulae derived in [5] are valid only under the assumption that all points on the planar patch are in front of the image plane before and after the motion (a fact which is explicitely stated in [5]). This is an artificial restriction which may not be true in a practical application e.g. in robotics.

Second, the sufficiency of a four-point match between the two images of the planar patch is presented as the result of an involved argument whereas it is really an immediate consequence of the fundamental theorem of projective geometry.

These two points illustrate the general weakness of computational (as opposed to conceptual) solutions to a problem: They tend to introduce unnecessary restrictions and they tend to obscure the underlying mathematical structure.

In this paper we present a conceptual solution to the moving plane problem which removes the restrictions of [5]. To emphasize the structural aspects of our solution, we present it in n-dimensions, as this introduces no extra difficulty. Since we want explicit motion formulae we must of course also handle some computations. The point is that we are able to introduce them at a later stage thereby retaining conceptual clarity as long as possible.

Mathematical formulation of the problem

Let us begin by establishing some notation. The real n-dimensional, euclidean vector space will be denoted by $\mathcal{R}^n$. Vectors will always be denoted by latin letters and scalars will always be denoted by greek letters. $[x_1, \ldots, x_m]$ denotes the linear span of the vectors $\{x_1, \ldots, x_m\}$, and $[x_1, \ldots, \hat{x}_i, \ldots, x_m]$ denotes the linear span of the vectors $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}$.

Let us now assume that we are "looking" at $(n+1)$ distinguishable points of a $(n-1)$-dimensional hyperplane π_x in $\mathcal{R}^n$ not containing the origin. This of course means that we are given $(n+1)$ distinct directions from the origin (eye) represented by the vectors $x_i \in \mathcal{R}^n, i = 1, \ldots, n+1$. We also know that for some positive scalar α_i, the point $\alpha_i x_i$ is in π_x. If we furthermore assume that the $(n+1)$ considered points are in general position, then we will be able to observe that $[x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}] = \mathcal{R}^n, \forall i$. Now suppose that the hyperplane is moved to a new position π_y (also not containing the origin) by some unknown isometry S of $\mathcal{R}^n$. Since the $(n+1)$ points are individually identifiable, to each old direction x_i there will correspond a new direction $y_i \in \mathcal{R}^n$ and to each old positive scalar α_i there will correspond a new positive scalar β_i such that the point $\beta_i y_i$ is in π_y. And since the old points $\alpha_i x_i$ were located in general position, the new points $\beta_i y_i$ will also be so situated which means that we will have $[y_1, \ldots, \hat{y}_i, \ldots, y_{n+1}] = \mathcal{R}^n, \forall i$.

The problem at hand can now be stated thus: Given the information presented above, to what extent can we determine the initial and final positions π_x and π_y of the moving hyperplane, i.e. what is the set of all motions

$$\{\mathcal{R}^n \ni \pi_x \overset{S}{\longmapsto} \pi_y \in \mathcal{R}^n\}$$

compatible with our observations?

Mathematically, this amounts to the following formulation:

(1) Given: vectors $x_i, y_i \in \mathcal{R}^n \quad , \quad i = 1, \ldots, n+1 \quad , \quad n \geq 3$
such that: $\forall i = 1, \ldots, n+1$

$$[x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}] = \mathcal{R}^n$$
$$[y_1, \ldots, \hat{y}_i, \ldots, y_{n+1}] = \mathcal{R}^n$$

Find: The collection of all
$\alpha_i > 0 \quad , \quad \beta_i > 0 \quad , \quad i = 1, \ldots, n+1$
U orthogonal $n \times n$ matrix with $\det U = 1$
$t \in \mathcal{R}^n$
$p \in \mathcal{R}^n$
such that: $\forall i = 1, \ldots, n+1$

(i) $\quad U\alpha_i x_i + t = \beta_i y_i$

(ii) $\quad p^T(\alpha_i x_i) = 1$

$\sim \cdot \sim$

Here we have used two well known facts:

i Each motion S can be uniquely decomposed into a rotation U around the origin followed by a translation t.

ii Since π_x does not contain the origin, its equation can be normalized to $p^T x = 1$.

From now on we will deal exclusively with the mathematical formulation (1) of our problem.

Note that this formulation does not assume the existence of a solution since no compatibility conditions are introduced between the $\{x_i\}$ and the $\{y_i\}$.

Reformulating the problem

Before trying to solve problem (1) it will be useful to subject it to some reformulation in order to deal with the main difficulty involved, which is the non-linearity of the relationship between $\alpha_i x_i$ and $\beta_i y_i$ as expressed in (1):(i).

First of all let us make the following observation:

(2) Under the prerequisites of problem (1) it follows directly from the fundamental theorem of projective geometry that there exists a unique (pair of) non-singular linear map(s) L (and $-L$): $\mathcal{R}^n \to \mathcal{R}^n$ with $|\det L| = 1$ such that $Lx_i \in [y_i]$.
Furthermore, a number of matching points which is smaller than $n+1$ does not determine this map whereas a larger number overdetermines it.

For more information on this subject see e.g. [1].

We can now formulate the following key lemma which relates the vectors $\alpha_i x_i$ and $\beta_i y_i$ in a *linear* way:

LEMMA:

(3) If α_i, β_i, U, t and p solve problem (1) and if L is the map described in (2) then there exists a unique $\lambda \in \mathcal{R}$ such that

$$\lambda L \alpha_i x_i = \beta_i y_i \quad , \quad i = 1, \ldots, n+1.$$

PROOF: see appendix.

This lemma will now allow us to reformulate problem (1) in terms of matrix calculus. By a series of suitable "coordinate transformations" the solutions to (1) will all be made to satisfy a certain matrix equation involving an unknown rank-1 perturbation of the identity, whose product with its own transpose is supposed to equal a known diagonal matrix. An independent scalar-valued condition on the desired perturbations will also be deduced, leading up to the equations expressed in (14).

Let $(x_1 \ldots \hat{x}_i \ldots x_{n+1})$ denote the $n \times n$ matrix whose column vectors are $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\}$ and let

(4) $$\varepsilon_i = \operatorname{sgn}(\det((x_1 \ldots \hat{x}_i \ldots x_{n+1})(y_1 \ldots \hat{y}_i \ldots y_{n+1})))$$

It is easily seen that a necessary condition of compatibility for the existence of a solution is given by the demand that

(5) $$\varepsilon_i = \varepsilon \quad \text{independently of } i \quad , \quad \forall i = 1, \ldots, n+1$$

which is a fact that can be checked a priori from the initial data.

By the previous lemma we know that if a solution exists we must have

$$\lambda L \alpha_i x_i = \beta_i y_i$$

But $\alpha_i > 0$, $\beta_i > 0$ which together with (4) and (5) implies that

$$\text{sgn}(\det \lambda L) = \varepsilon \tag{6}$$

To summarize:

> ε is known (from image data). L is calculated by (2) with $|\det L| = 1$. If n is odd we choose $\det L = \varepsilon$ and $\lambda > 0$ and if n is even we choose $\lambda > 0$ and $\det L = \varepsilon$ is given automatically.

Therefore we can without loss of generality assume that

$$\lambda > 0 \quad \text{and} \quad \det L = \varepsilon \tag{7}$$

By the lemma (3) and (*ii*) of (1) we now have

$$\begin{aligned}\lambda L \alpha_i x_i &= \beta_i y_i = U \alpha_i x_i + t = \\ &= U \alpha_i x_i + t p^T \alpha_i x_i = \\ &= (U + t p^T)(\alpha_i x_i) \quad , \quad i = 1, \ldots, n+1\end{aligned}$$

This gives

$$\begin{aligned}\lambda L &= U + t p^T = U(I + U^T t p^T) = \\ &= U(I + u p^T)\end{aligned} \tag{8}$$

where we have put $u = U^T t$.

Multiplying (8) with its transpose leads to

$$\lambda^2 L^T L = (I + p u^T)(I + u p^T) \tag{9}$$

Diagonalizing the symmetric, positive definite matrix $L^T L$:

$$\begin{aligned}&L^T L = V^T D V \\ &D = \begin{pmatrix} \delta_1^2 & & 0 \\ & \ddots & \\ 0 & & \delta_n^2 \end{pmatrix} \quad , \quad \delta_i \neq 0 \quad , \quad V^T V = I\end{aligned}$$

and substituting

$$\begin{aligned}a &= V p \\ b &= V u\end{aligned}$$

gives us back in (9):

$$\lambda^2 V^T D V = (I + V^T a b^T V)(I + V^T b a^T V) = \\ = V^T (I + a b^T)(I + b a^T) V \tag{10}$$

Multiplying (10) by V from the left and V^T from the right we arrive at the following equation:

$$\boxed{\lambda^2 D = (I + a b^T)(I + b a^T)} \tag{11}$$

where λ and D are known quantities and a and b are to be determined.

To get another equation connecting the unknown vectors a and b we can make use of the formula

$$\det(I + a b^T) = 1 + a^T b \tag{12}$$

which is easily established.

Now, since by (8):

$$\lambda L = U(I + u p^T) = U(I + V^T b a^T V) = \\ = U V^T (I + b a^T) V$$

taking determinant of both sides, we get from (7) and (12):

$$\lambda^n \varepsilon = 1 + a^T b \tag{13}$$

To sum up:

We have shown that solving problem (1) can be done by finding all vectors $a, b \in \mathcal{R}^n$ that satisfy the equations:

$$\begin{cases} (I + a b^T)(I + b a^T) = \lambda^2 D \\ 1 + a^T b = \lambda^n \varepsilon \end{cases} \tag{14}$$

where

$$\begin{cases} \lambda L = U + t p^T = U(I + u p^T) \quad , \quad u = U^T t \\ \varepsilon = \operatorname{sgn}(\det((x_1 \ldots \hat{x}_i \ldots x_{n+1})(y_1 \ldots \hat{y}_i \ldots y_{n+1}))) \\ \det L = \varepsilon \quad , \quad \lambda > 0 \\ L^T L = V^T D V \quad , \quad D = \begin{pmatrix} \delta_1^2 & & 0 \\ & \ddots & \\ 0 & & \delta_n^2 \end{pmatrix} \quad , \quad \delta_i \neq 0 \quad , \quad i = 1, \ldots, n \\ a = V p \quad , \quad b = V u \end{cases} \tag{15}$$

It is clear from the previous discussion that any solution to (1) must also solve (14) via the substitutions (15). Which of the solutions to (14) that are also solutions to (1) will be established explicitely in each separate case.

Computing solutions

Solving (14) in the general case of non-zero translation will be done by another reformulation (26) which displays the two dimensional character of our problem. (26) in turn will be solved by parametrizing it in a way which expresses the solutions as the four points of intersection between a circle and an ellipse (35).

Finally, by unwinding the sequence of substitutions involved, these points will be "decoded" and checked for compatibility with the constraints thus leaving two different solutions (modulo scaling) to problem (1) in the generic case.

The case of pure rotation

Since, by (1):(ii) we have $p \neq 0$ and since $a = Vp$, it follows that $a \neq 0$. Hence the only trivial solution to (14) is given by $b = 0$. In this case (14) reduces to

$$\begin{cases} \lambda^2 D = I \\ 1 = \lambda^n \varepsilon \end{cases} \Rightarrow \begin{cases} \lambda^2 \delta_i = 1 \quad , \quad \forall i = 1, \ldots, n \\ 1 = \lambda^n \varepsilon \end{cases}$$

Since $\varepsilon = \pm 1$ and $\lambda > 0$, it follows that $\varepsilon = \lambda = 1$ and hence

$$D = I$$

By (15) we then have

$$\begin{aligned} &L^T L = I \quad , \quad \text{i.e. } L \text{ is orthogonal} \\ &\det L = \varepsilon = 1 \\ &t = Uu = UV^T b = 0 \end{aligned}$$

Hence the motion S is a pure rotation

$$S = U$$

and from (15) it is given by

$$U = L$$

Conversely, if $L^T L = I, \det L = 1$ then it follows from (15) that

$$D = I \quad , \quad \varepsilon = 1 \quad , \quad \lambda = 1 \quad , \quad t = 0 \quad , \quad U = L$$

Hence the case of pure rotation corresponds exactly to the observable parameter-values:

(16) $$L^T L = I \quad , \quad \det L = 1$$

Note that p is completely undetermined which means that the initial position of the hyperplane π_x could be anywhere.

The case of non-zero translation

In this case we have $t \neq 0$, and we can without loss of generality assume that $|t| = 1$, which by (15) is equivalent to

$$|b| = 1 \tag{17}$$

To simplify the structure of equations (14) let us observe that if we take any vector $c \in \mathcal{R}^n$ such that $c \in [a, b]^\perp$ we have

$$\lambda^2 D c = c \tag{18}$$

which means that 1 is an eigenvalue of $\lambda^2 D$ with (algebraic) multiplicity $\geq \dim[a, b]^\perp \geq n - 2$. Hence we can renumber the eigenvectors of $\lambda^2 D$ so that

$$\lambda^2 D = \begin{pmatrix} \lambda^2\delta_1^2 & & & & 0 \\ & \lambda^2\delta_2^2 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \tag{19}$$

Furthermore we have

$$\det L^T L = 1 = \det D = \delta_1^2 \delta_2^2 (\delta^2)^{n-2} \tag{20}$$

where

$$\delta = \delta_3 = \cdots = \delta_n = \frac{1}{\lambda} \tag{21}$$

which means that

$$\delta_1 \delta_2 = \delta^{2-n} \tag{22}$$

Note that (19) implies that our problem is really 2-dimensional!

Also note that in $\mathcal{R}^n$ for $n > 3$ problem (1) has no solution in the generic case. By (19) we are demanding that $n - 2$ eigenvalues of $\lambda^2 D$ be equal which is a highly non-generic condition.

Now, by (19) we can choose a basis of $\mathcal{R}^n$ so that

$$a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{23}$$

Let us introduce new variables $v, w \in \mathcal{R}^2$ and a new 2×2 matrix $\tilde{D}$ by

$$v = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad , \quad w = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad , \quad \tilde{D} = \frac{1}{\delta^2} \begin{pmatrix} \delta_1^2 & 0 \\ 0 & \delta_2^2 \end{pmatrix} \tag{24}$$

and observe that (17) and (23) imply

$$|w| = 1 \tag{25}$$

Summing up, we see that solving (14) is equivalent to solving the equations

$$\begin{array}{ll} (i) & \\ (ii) & \\ (iii) & \end{array} \begin{cases} (I + vw^T)(I + wv^T) = \tilde{D} \\ 1 + v^T w = \varepsilon\delta^{-n} \\ |w| = 1 \end{cases} \tag{26}$$

for the unknown vectors $v, w \in \mathcal{R}^2$. (ε, δ and $\tilde{D}$ are known from image data.)

To solve (26) we will make use of the so called *Sherman-Morrison formula:*

$$(I + vw^T)^{-1} = I - \frac{1}{1 + v^T w} vw^T \tag{27}$$

Remembering that $1/\varepsilon = \varepsilon$, we get from (27) and (26):(ii)

$$(I + vw^T)^{-1} = I - \varepsilon\delta^n vw^T \tag{28}$$

Note that by (26):(ii) this inverse will always exist.

Multiplying (26):(i) by $w^T(I - vw^T)^{-1}$ from the left gives us

$$w^T + v^T = (w^T - \varepsilon\delta^n w^T vw^T)\tilde{D} \tag{29}$$

and transposing (29) leads to

$$w + v = \varepsilon\delta^n \tilde{D} w \tag{30}$$

where we have made use of the fact that

$$1 - \varepsilon\delta^n v^T w = 1 - \varepsilon\delta^n(\varepsilon\delta^{-n} - 1) = \varepsilon\delta^n$$

Hence, from (30) we can express v in terms of w:

$$v = (\varepsilon\delta^n \tilde{D} - I)w \tag{31}$$

Plugging (31) into (26):(ii) and making use of the indentities $w^T w = 1$ and $\varepsilon^2 = 1$ we get

$$w^T \delta^{2n} \tilde{D} w = 1 \tag{32}$$

Letting

$$\begin{cases} \alpha_1^2 = \delta^{2n-2}\delta_1^2 = \{\text{by } (22)\} = \dfrac{\delta^2}{\delta_2^2} \\[2ex] \alpha_2^2 = \delta^{2n-2}\delta_2^2 = \{\text{by } (22)\} = \dfrac{\delta^2}{\delta_1^2} \end{cases} \tag{33}$$

we finally end up with the equation

$$w^T \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix} w = 1 \tag{34}$$

Hence we have transformed (26) into the system

$$\begin{cases} \alpha_1^2 w_1^2 + \alpha_2^2 w_2^2 = 1 \\ w_1^2 + w_2^2 = 1 \end{cases} \tag{35}$$

where $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is unknown and α_1 and α_2 are given by (33).

This is the ellipse-circle-intersection formulation that we promised earlier.

Classifying solutions

We are now finally in a position to express the solution of problem (1) through (33), (26) and (14). Before doing this let us observe that by (15), (27) and (14) we have

$$\begin{aligned} U &= \lambda L(I + up^T)^{-1} = \\ &= \lambda L(I - \frac{1}{1 + p^T u} up^T) = \\ &= \lambda L(I - \frac{1}{1 + a^T b} up^T) = \\ &= \lambda L(I - \frac{1}{\lambda^n \varepsilon} up^T) \end{aligned} \tag{36}$$

Hence we can collect the relevant solution formulae in terms of a and b:

$$\begin{cases} p = V^T a \\ u = V^T b \\ U = \lambda L(I - \frac{1}{\lambda^n \varepsilon} up^T) \\ t = Uu \end{cases} \tag{37}$$

where λ, ε, L and V are computed directly from image data.

Solving (35) leads to six different cases:

I $\quad (\alpha_1^2 = 1 \ , \ \alpha_2^2 \neq 1)$:

In this case we immediately get

$$w_2 = 0 \quad , \quad w_1^2 = 1$$

Hence

$$w = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \psi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\psi = +1$ or -1.

Now, by (31)

$$\begin{aligned} v &= \psi(\varepsilon\delta^{n-2}\delta_1^2 - 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \{\text{by } (33)\} = \\ &= \psi(\varepsilon\delta^{-n}\alpha_1^2 - 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \psi(\varepsilon\delta^{-n} - 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Unwinding our substitutions we get from (23) and (24)

$$a = \psi(\varepsilon\delta^{-n} - 1)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad b = \psi \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{38}$$

But from (15) we have

$$p = V^T a \tag{39}$$

and by (1):(*ii*) we have

$$p^T x_1 > 0 \tag{40}$$

where x_1 is known from image data.

Hence ψ is determined and thus also a and b. Substituting (38) into (37) therefore gives the unique solution to problem (1) in this case.

II $(\alpha_1^2 \neq 1 \ , \ \alpha_2^2 = 1)$:

This case is reduced to case I by interchanging index 1 and 2.

III $(\alpha_1^2 = \alpha_2^2 = 1)$:

In this case we get from (35) that all w with $|w| = 1$ are solutions.

Hence by (33)

$$\delta_1 = \delta_2 = \delta \tag{41}$$

and by (31)

$$\begin{aligned} v &= \left(\varepsilon\delta^{n-2}\begin{pmatrix} \delta^2 & 0 \\ 0 & \delta^2 \end{pmatrix} - I\right) w = \\ &= (\varepsilon\delta^n - 1) w \end{aligned} \tag{42}$$

Hence we must have

$$\det(I + vw^T) = 1 + v^T w = 1 + \varepsilon\delta^n - 1 = \varepsilon\delta^n$$

But according to (26):

$$\det(I + vw^T) = \varepsilon\delta^{-n}$$

Hence

$$\varepsilon\delta^n = \varepsilon\delta^{-n}$$
$$\delta^2 = 1$$

and since $\delta > 0$ we conclude that $\delta = 1$.
Thus we can rewrite (42)

$$v = (\varepsilon - 1)w$$

and by (23) and (24) we have the solutions

$$a = (\varepsilon - 1)b \quad , \quad b = \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad |w| = 1 \tag{43}$$

Now, since by (15)

$$p = V^T a = V^T(\varepsilon - 1)b \tag{44}$$

and since by (1):(*ii*) $p \neq 0$, we must have

$$\varepsilon = -1 \tag{45}$$

Hence by (44)

$$p = -2V^T b$$

and since $u = V^T b$ we get from (8):

$$\lambda L = U(I + V^T b b^T V(-2)) = UV^T(I - 2bb^T)V \tag{46}$$

But since by (17) $|b| = 1$, $(I - 2bb^T)$ represents a reflection in $[b]^\perp$. Hence

$$(I - 2bb^T)^{-1} = (I - 2bb^T)$$

and we get from (46):

$$U = \lambda L V^T(I - 2bb^T)V \tag{47}$$

Note that since U is orthogonal, and since by (15) and (45) $\det L = -1$, it follows from (47) that

$$\begin{cases} \lambda = 1 \\ L \text{ is orthogonal with } \det L = -1 \end{cases} \tag{48}$$

Hence (48) is an a priori condition for case III.

IV $(\alpha_1^2 > 1\ ,\ \alpha_2^2 > 1)$:

This immediately implies

$$w_1 = w_2 = 0$$

which violates (35).

Hence there is no solution in this case.

V $(\alpha_1^2 < 1\ ,\ \alpha_2^2 > 1)$:

Same conclusion as in case IV.

VI $(\alpha_1^2 < 1 < \alpha_2^2)$:

This is the generic case. It gives a priori four solutions for w:

$$w_1 = \pm\sqrt{\frac{\alpha_2^2 - 1}{\alpha_2^2 - \alpha_1^2}}\quad ,\quad w_2 = \pm\sqrt{\frac{1-\alpha_1^2}{\alpha_2^2 - \alpha_1^2}} \tag{49}$$

and by (31) each one of these gives a solution for v.

Letting $w_1 > 0, w_2 > 0$ in (49) we can collect our solution-candidates

$$\begin{array}{cccccc} & & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \\ w & = & \left|\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right| & \begin{pmatrix} w_1 \\ -w_2 \end{pmatrix}\Big| & \begin{pmatrix} -w_1 \\ w_2 \end{pmatrix}\Big| & \begin{pmatrix} -w_1 \\ -w_2 \end{pmatrix}\Big| \\ v & = & \left|\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right| & \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}\Big| & \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}\Big| & \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}\Big| \end{array}$$

Note that these four cases can each be expressed by multiplying the vectors of case *1* by respectively the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{50}$$

Denoting by S an arbitrary matrix from this set, we observe that if v, w are solutions to (26) we have

$$\begin{aligned}(I + Sv(Sw)^T)(I + Sw(Sv)^T) &= S(I + vw^T)(I + wv^T)S = \\ &= SDS = DSS = D\end{aligned}$$

and

$$1 + (Sv)^T(Sw) = 1 + v^T S^T S w = 1 + v^T w$$

as well as

$$|Sw| = |w| = 1$$

Hence the four solutions to (35) are also solutions to (26) and by analogous calculations also to (14).

Let w denote one of these solutions, e.g. the one with $w_1 > 0, w_2 > 0$ in (49), and let v denote the corresponding solution for v.

All solutions to (26) are then of the form Sv, Sw, and letting

$$(51) \qquad a = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad S \leftarrow \begin{pmatrix} S & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

all solutions to (14) are of the form Sa, Sb, where S is one of the matrices corresponding to (50) via (51).

Going back to the original problem (1) and collecting the relevant expressions we have the four different solution-candidates

$$\begin{cases} p = V^T S a \\ u = V^T S b \\ t = U u \\ U = \lambda L(I + up^T)^{-1} = \lambda L(I + V^T S b a^T S V)^{-1} \end{cases}$$

But since the four S-matrices of (50) are pairwise negations of each other and since by (39) and (40) we cannot have both p and $-p$ as a solution to (1), two of the candidates can be ruled out by the simple sign check (40) in the image, thus leaving two solutions to problem (1) in the general case.

Conclusion

We have presented a general solution to the moving hyperplane problem which emphasizes the structural aspects but which is also suitable for computations. This has been achieved by combining the conceptual power of projective geometry with the computational power of matrix calculus. We have chosen not to present our solution formulae as explicit expressions in the original variables in favour of a more algorithmic approach in order to facilitate their translation into computer programs.