On the Use of the Double Algebra in Computer Vision

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Abstract

The purpose of this paper is to introduce the basic concepts of the Double Algebra and illustrate its potential use in Computer Vision. We give some very simple examples of projective invariants, i.e. quantities which do not depend on the specific coordinate system and which are invariant under projective transformations. We show how the Double Algebra provides a natural algebra for projective geometry, which is much closer to synthetic geometry than the usual coordinate algebra. Roughly speaking, it is an algebra for the meet and join operations in projective geometry. It can be used to list all projective invariants on a finite number of points and lines in the projective plane.

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1 Introduction

The purpose of this paper is to introduce the basic concepts of the Double Algebra and to illustrate its potential use in Computer Vision and Robotics. The double Algebra is the natural framework for projective geometry. Roughly, it is an algebra for the meet and join operations of projective geometry which is much closer to synthetic geometry than ordinary coordinate algebra. After a short historic background we will construct the algebra and state the main results. No proofs will be given, but they can be found in the references. We will show, through some examples, how it can be used to find all projective invariants for finite point–line configurations in the projective plane. Examples are also given of projective invariants on planar curves. We strongly believe that this beautiful piece of mathematics could provide a powerful tool in Computer Vision and Robotics.

2 History and Background

Geometry, according to Klein, deals purely with properties invariant under some group of transformations. However, in most of the applications geometric problems are formulated in coordinates with respect to some basis. This leads quite often to polynomial equations in several variables, that seldom admit treatment by elementary methods. Sometimes, when there are only a few variables, these polynomial equations can be handled using the powerful tools derived from the theory of Gröbner bases. There are two obvious drawbacks with this approach. Firstly, we introduce non-invariant expressions, and secondly the polynomials, especially those of the Gröbner basis, do not possess a natural interpretation in geometrical terms.

Hilbert showed–Hilbert's Finiteness Theorem– that, under certain mild assumptions on the group of transformations, there always exists a finite number of invariant polynomials –a complete set of invariants– such that all invariant polynomials can be expressed as polynomials in these.

Let V be a vector space over some field K with a basis $\{e_1, \ldots, e_n\}$, and let $x_i \in K^n$ be the coordinates for $v_i \in V$ with respect to this basis. The determinant of the $n \times n$ matrix $(x_1, \ldots, x_n)$ is denoted by $[v_1, \ldots, v_n]$ and is called a **bracket** (it certainly depends on the basis). If $T : V \mapsto V$ is linear we see that

$$[Tv_1, \ldots, Tv_n] = det(T)[v_1, \ldots, v_n]$$

Thus $[v_1, \ldots, v_n]$ is an invariant under the special linear group on V.

Projective Geometry deals with properties which are invariant under the Special Linear group, i.e. linear transformations with determinant one. The First Fundamental Theorem of Projective Geometry states that these brackets constitute a complete set of invariants. If m vectors $v_1, \ldots, v_m$ are given, and $m \geq n$, there are $\binom{m}{n}$ of them. In "bracket algebra" all brackets are treated as formal variables between which there are certain algebraic relations –the so-called Plücker-Grassmann syzygies. There is a famous algorithm –the straightening algorithm– by which all bracket polynomials can be written in a normal form. Recently (1990), Bernt Sturmfels showed that the straightening algorithm is in fact a normal form algorithm with respect to a certain explicit Gröbner base for the Plücker-Grassmann ideal. The polynomials in this Gröbner basis is the so-called Van der Waerden syzygies and the monomial order is the Tableau order.

In the late 19^{th} century, mathematicians tried to develop an algebra of n–dimensional space, analogously to the representation of R^2 with complex numbers, in which geometric operations could be carried out using purely algebraic operations. In 1844, Hermann Grassmann's "Die lineale Ausdehnungslehre" was published. There, he introduced an algebra for the **join** and **meet** operations of projective geometry. Unfortunately, his work was not fully understood until recently, when Marilena Barabei, Andrea Brini and Gian-Carlo Rota in 1985, wrote an article "On the Exterior Calculus of Invariant Theory", Journal of Algebra 96, pp 120-160. They formulated Grassmann's ideas in a modern mathematical terminology, which give a natural framework for projective geometry.

3 Peano Spaces

The Italian mathematician Peano was probably the first to realize the deep importance of brackets (G. Peano, "Calcolo Geometrico Secondo l'Ausdehnungslehre di H. Grassmann", Fratelli Bocca Editori, Torino, 1888.)
Definition: A Peano space is an n-dimensional vector space over a field K, together with a non-degenerate, alternating, n-linear form $[\quad]$ called the bracket.

Thus

1. $[\quad]: V^n \to K, \qquad x_1, \ldots, x_n \longmapsto [x_1, \ldots, x_n]$
2. $[x_1, \ldots, x_n]$ is linear in each variable.
3. $[x_1, \ldots, x_n] = 0$ if two of the x_i's are equal.
4. $[x_1, \ldots, x_n] \neq 0$ for some $x_1, \ldots, x_n \in V^n$.

Let $e = e_1, \ldots, e_n$ be some basis for V and let $y_i \in K^n$ be the coordinates of $x_i \in V$ with respect to e. Then

$$[x_1, \ldots, x_n] = det(y_1, \ldots, y_n)$$

is a bracket on V.

If A is an $n \times n$ matrix, then

$$det(Ay_1, \ldots, Ay_n) = det(A)det(y_1, \ldots, y_n)$$

Hence $[\quad]$ is invariant under the Special Linear Group on V of linear transformations with determinant one. It is easy to see that **all** brackets are of the above form for some basis.

3.1 Some examples on the use of bracket in $P^1(R)$ and $P^2(R)$.

Here, we will assume that the reader is familiar with elementary projective geometry.

1. **The double ratio in $P^1(R)$.**

   Consider four points x_1, x_2, x_3, x_4 on $P^1(R)$. As usual, we consider them as nonzero points in R^2 (homogeneous coordinates). Define

   $$D(x_1, x_2, x_3, x_4) = \frac{[x_1, x_3][x_2, x_4]}{[x_1, x_4][x_2, x_3]}$$

   where $[\quad]$ is some bracket on R^2. We note that

- $D(Tx_1, Tx_2, Tx_3, Tx_4) = D(x_1, x_2, x_3, x_4)$ for all invertible $T : R^2 \mapsto R^2$ and
- $D(\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \alpha_4 x_4) = D(x_1, x_2, x_3, x_4)$ for all nonzero α_i's in R.

D is in fact the well-known double ratio $(x_1 x_3 : x_4 x_2)$

2. **An invariant on six points in $P^2(R)$**

Let $x_1, \ldots, x_6$ be six points in $P^2(R)$ considered as six nonzero vectors in R^3. Suppose also that, for some arbitrary choice of bracket in R^3,

$$[x_2, x_3, x_5] \neq 0 \neq [x_1, x_4, x_6]$$

Then

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{[x_1, x_3, x_5][x_2, x_4, x_6]}{[x_2, x_3, x_5][x_1, x_4, x_6]}$$

is an invariant under all invertible linear transformations on R^3. Moreover, F is invariant under scaling i.e.

$$F(\alpha_1 x_1, \ldots, \alpha_6 x_6) = F(x_1, \ldots, x_6)$$

for all nonzero a_i's in R.

3. **An invariant on five points in $P^2(R)$.**

Put $x_5 = x_6$ above and define $G(x_1, x_2, x_3, x_4, x_5) = F(x_1, x_2, x_3, x_4, x_5, x_5)$ which is clearly invariant.

4. **A projective invariant on a smooth non-convex curve Γ in the plane.**

Consider a planar curve as in figure 1:

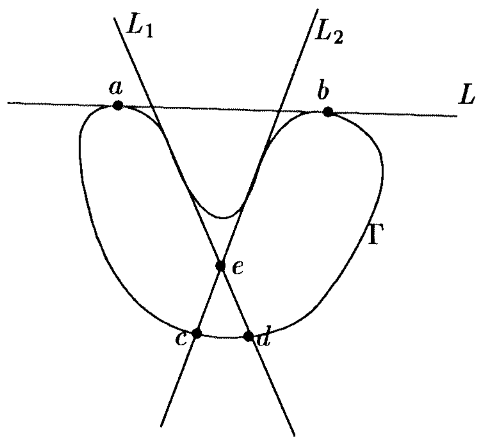


Figure 1: A projective invariant on a smooth curve Γ

Draw the double tangent L and the two inflexion lines L_1 and L_2. The points a, b, c, d, e are constructed as indicated in the figure. Then

$$\frac{[a, b, c][d, e, a]}{[a, b, d][c, e, a]}$$

is a projective invariant on Γ.

5. **The quadric through five points in $R^2(R)$.**

Let $x_1, \ldots, x_5$ be five points in $P^2(R)$ and put for $x \in R^2(R)$

$$E(x) = [x_1, x_3, x_5][x_2, x_4, x_5][x_2, x_3, x][x_1, x_4, x] - [x_2, x_3, x_5][x_1, x_4, x_5][x_1, x_3, x][x_2, x_4, x]$$

Then $E(x)$ is a homogeneous nonzero polynomial of degree two such that

$$E(x_1) = E(x_2) = E(x_3) = E(x_4) = E(x_5) = 0$$

6. **The double ratio of 4 points on a line in $P^2(R)$.**

Let x_1, x_2, x_3 and x_4 be points on a line in $P^2(R)$ not containing x_5. Then one easily verifies that $G(x_1, x_2, x_3, x_4, x_5)$, where G is as in the third example, is independent of x_5. Therefore, it is the double ratio of four points on a line in $P^2(R)$.

7. **A projective invariant on two disjoint planar sets M and N in $P^2(R)$.**

The points a, b, c, d, e are determined as it is indicated in the figure below:

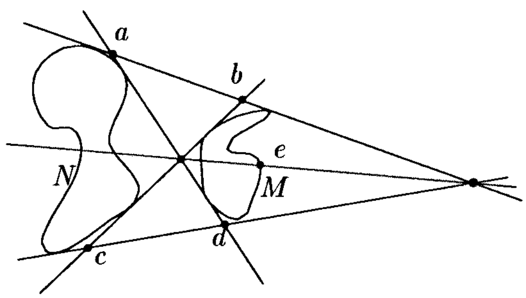


Figure 2: A projective invariant on two disjoint planar sets M and N.

Then

$$\frac{[a, b, c][d, e, a]}{[a, b, d][c, e, a]}$$

is a projective invariant.

Remark: The above examples are just very simple illustrations of how easy it is to find projective invariants in an almost automatic manner. Note that every expression of the form

$$\frac{[\cdot, \ldots, \cdot] \cdots [\cdot, \ldots, \cdot]}{[\cdot, \ldots, \cdot] \cdots [\cdot, \ldots, \cdot]}$$

in which every vector, denoted by a dot inside the brackets, occurs the same number of times in the numerator and the denominator, is a projective invariant. What is extremely important is that **every** invariant can be built up by combining expressions of this form. This means that we can list them all.

4 Projective Geometry and the Double Algebra.

In this section we will in detail construct the Double Algebra, essentially as suggested in the paper of Rota et al. Though no proofs are given, the mathematically sophisticated reader should have no difficulties in doing them as exercises. We conclude the section with some examples showing how close the Double Algebra is to Synthetic Geometry and how easy it is to use.

Let V be a vector space over a field K of dimension n. The projective geometry on V is the set $P(V)$ of all subspaces of V.

The **join** $A \vee B$ of two subspaces A and B is the least subspace of V containing both A and B. Hence,

$$A \vee B = A + B = \{a + b : a \in A, b \in B\}$$

The **meet** $A \wedge B$ of two subspaces A and B is the largest subspace of V contained in both A and B. Hence,

$$A \wedge B = A \cap B$$

These operations turn $P(V)$ into a lattice.

The double algebra $G(V)$, that we shall construct below, will be a vector space over K endowed with two algebraic operations –deliberately, called meet and join and denoted by the same symbols $\vee$ and $\wedge$ – which correspond to the meet and join operations on $P(V)$ in a very nice way. Some objects in $G(V)$, called **extensors,** will correspond to well defined subspaces in $P(V)$. In fact, if the set of extensors is denoted by $E(V)$ we will define a mapping $E(V) \to P(V)$, $E(V) \ni X \mapsto \overline{X} \in P(V)$ such that if X and Y are in $E(V)$ we have

$$\overline{X \vee Y} = \overline{X} \vee \overline{Y} \quad \text{if} \quad \overline{X} \wedge \overline{Y} = 0 \qquad \text{and} \qquad \overline{X \wedge Y} = \overline{X} \wedge \overline{Y} \quad \text{if} \quad \overline{X} \vee \overline{Y} = V$$

Philosophically, $G(V)$ can be regarded as an algebraic background against which $P(V)$ can be seen more clearly. Proofs of statements in $P(V)$ are easier to do in $G(V)$ with its richer structure. The price we have to pay is that not all objects in $G(V)$ can be interpreted in $P(V)$.

4.1 The construction of $G(V)$.

As before, we let V be a Peano space of dimension n with a bracket [ ]. The following statement is easily verified:

$$[x_1, \ldots, x_n] = 0 \Longleftrightarrow \{x_1, \ldots, x_n\} \qquad \text{are linearly dependent.}$$

Elements $x_1 \ldots x_k$ in $V^k = V \times \cdots \times V$ are called **monomials** of degree k. If $X = x_1 \ldots x_k$ and $Y = y_1 \ldots y_\ell$ are monomials, their product XY is the monomial $Z = XY = x_1 \ldots x_k y_1 \ldots y_\ell$ in $V^{k+\ell}$. This product extends to the set

$$S(V) = \{\sum_i \alpha_i m_i : \alpha_i \in K \quad \text{and } m_i \text{ is a monomial}\}$$

of formal finite linear combinations of monomials, in the following distributive manner

$$\left(\sum_i \alpha_i m_i\right)\left(\sum_j \beta_j w_j\right) = \sum_{i,j} \alpha_i \beta_j m_i w_j$$

These definitions turn $S(V)$ into the free associative algebra generated by V. Elements in the set $S_d(V) = \{\sum \alpha_i m_i : \alpha_i \in K, \quad m_i \in V^d\}$ are called homogeneous of degree d.

We extend the domain of the bracket to $S_n(V)$ by

$$[\sum \alpha_i m_i] = \sum \alpha_i [m_i]$$

We now introduce an equivalence relation $\sim$ on $S(V)$ by the following reduction rules:

- $X \sim 0$ if $X \in S_k(V)$ and $k > n$
- $X \sim 0$ if $X \in S_n(V)$ and $[X] = 0$
- $X \sim 0$ if $[XY] = 0$ for all monomials Y such that $XY \in S_n(V)$.

These reduction rules mean that monomials $x_1 x_2 \cdots x_s$, where the vectors $x_1, x_2, \ldots, x_s$ are linearly dependent are considered to be zero. Moreover, if x and y are vectors in V then xy and $-yx$ are considered equivalent.

The exterior algebra $G(V)$ on V is now defined as $S(V)$ together with the reduction rules above, i.e.

$$G(V) = S(V)/\sim$$

The product that $G(V)$ inherits from $S(V)$ we denote by $\vee$ and call it the join. The set of monomials $E(V)$ in $G(V)$ are called extensors and those of degree k, k–extensors. The set of k–extensors is denoted by $E_k(V)$. Hence,

$$E_k(V) = V^k/\sim \qquad E(V) = \cup_k E_k(V)$$

The reader should have no difficulty in verifying the following:

1. If $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are bases of the same subspace, then

$$x_1 \vee \ldots \vee x_k = c \cdot y_1 \vee \ldots \vee y_k \quad \text{for some} \quad c \in K.$$

2. $x_1 \vee \ldots \vee x_k = 0$ if and only if $\{x_1, \ldots, x_k\}$ are linearly dependent.

3. If $x_1 \vee \ldots \vee x_k = c \cdot y_1 \vee \ldots \vee y_k \neq 0$, then $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are bases of the same subspace.

If $X = x_1 \vee \ldots \vee x_k \in E_k(V)$, then the linear span of $\{x_1, \ldots, x_k\}$ is denoted by $\overline{X}$. We thus have a mapping

$$E_k(V) \longrightarrow P(V)$$

such that if $X \in E_k(V) \backslash 0$ then $\overline{X} \in P_k(V)$, where $P_k(V)$ is the set of k-dimensional subspaces of V.

Theorem 1: Let X and Y be extensors, then

$$\overline{X \vee Y} = \overline{X} \vee \overline{Y} \qquad \text{if} \qquad \overline{X} \wedge \overline{Y} = 0$$

Note again, that we use the same symbols for the join and the meet in $G(V)$ and $P(V)$.

In order to define the meet operation on $G(V)$ we need some further notation.

Let $s < t$ be positive integers and put

$$\Lambda(t,s) = \{\lambda = (\lambda_1, \ldots, \lambda_s) : 1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq t\}$$

The complement λ^c of $\lambda \in \Lambda(t,s)$ is the element of $\Lambda(t,t-s)$ such that

$$\lambda \cup \lambda^c = \{\lambda_1, \ldots, \lambda_s, \lambda_1^c, \ldots, \lambda_{t-s}^c\} = \{1, 2, \ldots, t\}$$

The sign of the permutation $(\lambda_1, \ldots, \lambda_s, \lambda_1^c, \ldots, \lambda_{t-s}^c)$ of $(1, 2, \ldots, t)$ is denoted by $sign(\lambda, \lambda^c)$. If $X = x_1 \vee \cdots \vee x_t \in E_t(V)$ we define for $\lambda \in \Lambda(t,s)$

$$X_\lambda = x_{\lambda_1} \vee \cdots \vee x_{\lambda_s} \in E_s(V)$$

At last, we can define the meet $X \wedge Y$ of $X \in E_k(V)$ and $Y \in E_\ell(V)$, provided that $k + \ell \geq n$ by

Definition: $X \wedge Y = \sum sign(\lambda, \lambda^c)[X_\lambda \vee Y]X_{\lambda^c}$

where the sum is taken over all $\lambda \in \Lambda(k, n-\ell)$.

It is not obvious, but $X \wedge Y$ is in fact an extensor in $E_{k+\ell-n}(V)$.

Remark: There is another equivalent definition of the meet, namely

$$X \wedge Y = \sum sign(\lambda, \lambda^c)[X \vee Y_{\lambda^c}]Y_\lambda$$

where the sum is taken over all $\lambda \in \Lambda(\ell, k+\ell-n)$. In case $k + \ell < n$ we put $X \wedge Y = 0$.

The meet and join operations are extended distributively to all of $G(V)$. The geometric meaning of the meet is clarified by the next

Theorem 2: Let X and Y be extensors, then

$$\overline{X \wedge Y} = \overline{X} \wedge \overline{Y} \qquad \text{if} \qquad \overline{X} \vee \overline{Y} = V$$

Observe also that every linear operator $T : V \mapsto V$ extends in a canonical manner to $G(V)$ by:

$$T(x_1 \vee \ldots \vee x_k) = Tx_1 \vee \ldots \vee Tx_k$$

It follows that for extensors X and Y

$$T(X \vee Y) = TX \vee TY \qquad \text{and} \qquad T(X \wedge Y) = TX \wedge TY$$

To be able to do calculations in $G(V)$ we introduce a basis $e = \{e_1, \ldots, e_n\}$ for V. Put $E = e_1 \vee e_2 \vee \ldots \vee e_n$ and let $X \in E_k(V)$. Then $[X \vee E_\lambda]$, where $\lambda \in \Lambda(n, n-k)$, is called the λ-coordinate for X. There are $\binom{n}{n-k} = \binom{n}{k}$ such coordinates and they are called the Plücker–Grassmann coordinates relative to the basis e.

To summarize: We have constructed an algebra $G(V)$ with two products called the meet $\wedge$ and join $\vee$, which correspond to the meet and join operations in the projective geometry $P(V)$. Subspaces of dimension k in $P(V)$ correspond to k–extensors in $G(V)$. Two nonzero extensors correspond to the same subspace if and only if they are proportional.

4.2 Some Examples

We give some simple examples of the practical use of the meet and join products in planar projective geometry. This list could of course be made much longer and also include higher dimensions. These examples illustrate how easy it is to construct invariants. In fact, to any finite configuration of points and lines in $P^2(R)$, it is almost trivial to write down the list of all projective invariants for the configuration.

1. Suppose two lines A and B and two points a and b are given generically in $P^2(R)$. Then

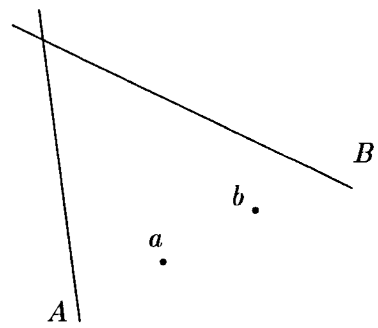


Figure 3: A projective invariant for a pair of lines and a pair of points.

the expression

$$\frac{[A \vee a][B \vee b]}{[A \vee b][B \vee a]}$$

is a projective invariant for the configuration.

2. Consider three pairs of points in $P^2(R)$, see figure 3. We want to express in terms of

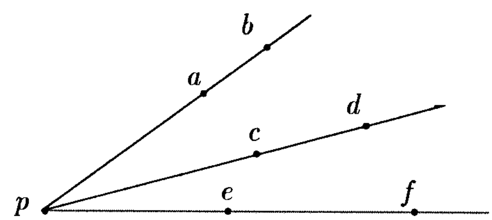


Figure 4: Three lines passing through the same point

the double algebra that the lines joining these three pairs pass through the same point p. The line joining a with b is $a \vee b$ and similarly for $c \vee d$, $e \vee f$. The intersection of $a \vee b$ with $c \vee d$ is

$$p = (a \vee b) \wedge (c \vee d) = [a, c, d]b - [b, c, d,]a$$

The point p lies on $e \vee f$ precisely when $[p, e, f] = 0$. Hence, we get

$$[p, e, f] = [a, c, d][b, e, f] - [b, c, d][a, e, f] = 0$$

Hence, the lines $a \vee b$, $c \vee d$ and $e \vee f$ meet in a point in $P^2(R)$ if and only if

$$[a, c, d][b, e, f] = [b, c, d][a, e, f]$$

3. Formulation of the well known five point problem in Computer Vision:

Suppose that five points a, b, c, d, e in R^3 are moved rigidly to new positions a', b', c', d', e'. If the motion were a pure translation the five lines joining a with a',...,e with e' would be parallel. We therefore see that for some 3×3 rotation matrix U, the five two-dimensional subspaces

$$\overline{a \vee Ua'} \quad \overline{b \vee Ub'} \quad \overline{c \vee Uc'} \quad \overline{d \vee Ud'} \quad \overline{e \vee Ue'}$$

intersect in a one-dimensional subspace $\bar{t}$. (Note that t is the translation direction and U the rotation part of the motion.) By the example above we, thus, have

$$[a,b,Ub'][Ua',x,Ux'] = [Ua',b,Ub'][a,x,Ux'] \qquad \text{where} \qquad x = c,d,e$$

We can rewrite this as

$$[a,b,Ub'][a',U^T x,x'] = [a',U^T b,b'][a,x,Ux'] \qquad \text{where} \qquad x = c,d,e$$

Note that the above equations are homogeneous in $a, a', \ldots, e, e'$ and U. A well-known representation of the 3×3 rotation matrices, for which -1 is not an eigenvalue is

$$U = \frac{1}{1+u_1^2+u_2^2+u_3^2}\begin{bmatrix} 1+u_1^2-u_2^2-u_3^2 & 2(u_1u_2-u_3) & 2(u_1u_3+u_2) \\ 2(u_1u_2+u_3) & 1+u_2^2-u_1^2-u_3^2 & 2(u_2u_3-u_1) \\ 2(u_1u_3-u_2) & 2(u_2u_3+u_1) & 1+u_3^2-u_1^2-u_2^2 \end{bmatrix}$$

For given values of the projections of a,b,c,d and e we get three polynomial equations in u_1,u_2,u_3 of degree 4. It took less than three minutes to calculate a Gröbner basis for this system (in degree reverse lexicographic order). In a future paper we will use a recent algorithm by Donald Pedersen to calculate the exact number of real solutions of these equations, which moreover satisfy certain further inequalities arising from physical conditions of the problem.

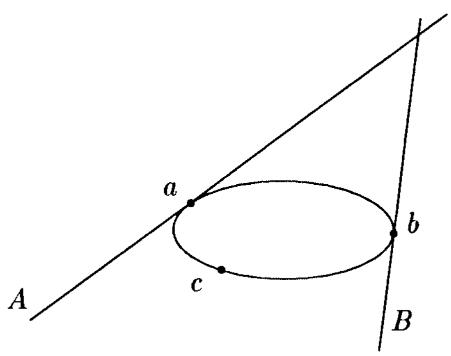
4. As we saw, the equation of a quadric through the points a,b,c,d,e is given by

$$E(x) = [a,c,e][b,d,e][b,c,x][a,d,x] - [b,c,e][a,d,e][a,c,x][b,d,x] = 0$$

The tangent line at a is obtained by taking the derivative with respect to x

$$T_a = [a,c,e][b,d,e][b,c,a](a \vee d) - [b,c,e][a,d,e][b,d,a](a \vee c)$$

5. Find the equation of the quadric in $P^2(R)$ that has tangent A in a, tangent B in b and passes through the point c.



Here A and B are lines or 2-extensors and a, b, c are points or 1-extensors, such that $A \vee a = B \vee b = 0$. It can be verified that the homogeneous polynomial

$$E(x) = [a \vee b \vee c]^2[A \vee x][B \vee x] - [a \vee b \vee x]^2[A \vee c][B \vee c]$$

satisfies our requirements. Clearly $E(a) = E(b) = E(c) = 0$. Taking derivatives, the tangent to $E = 0$ at a is found to be

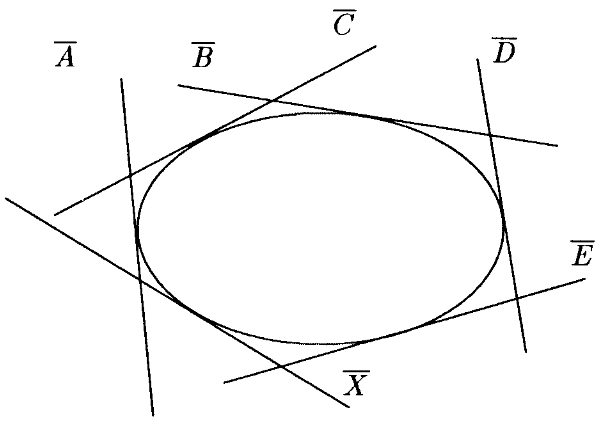
$$T_a = [a \vee b \vee c]^2[B \vee a]A = \text{constant} \cdot A$$

and the tangent at b is

$$T_b = [a \vee b \vee c]^2[A \vee b]B = \text{constant} \cdot B$$

which was what we wanted.

6. The quadric, in line–coordinates, tangent to five given lines.



Given five lines in $P^2(R), \overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}$ where A, B, C, D and E are 2-extensors in R^3, i.e. $A, B, C, D, E \in E_2(R^3)$. Then $A \wedge B \wedge C$, $A \wedge B \wedge D$, e.t.c. are 0-extensors, that is real numbers. It is easily verified that

$$(A \wedge C \wedge E)(B \wedge D \wedge E)(B \wedge C \wedge X)(A \wedge D \wedge X) = (B \wedge C \wedge E)(A \wedge D \wedge E)(A \wedge C \wedge X)(B \wedge D \wedge X)$$

if and only if $\overline{X}$ is a line tangent to the quadric tangent to the five lines $\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}$. By introducing some basis we get the quadric in line–coordinates.

7. A projective invariant on six lines in $P^2(R)$.

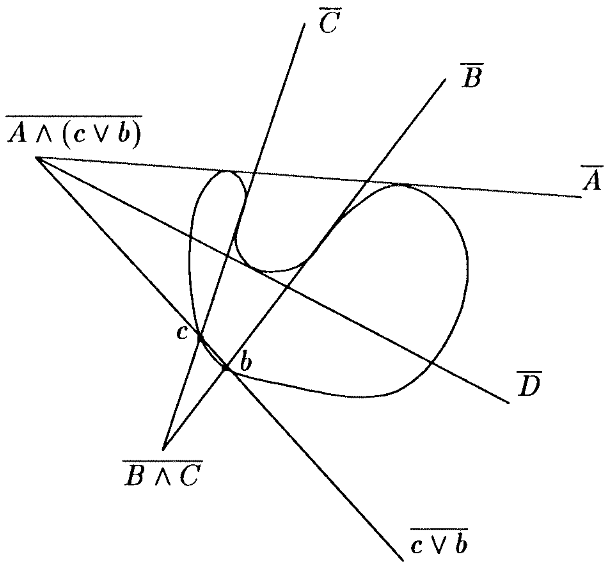
Let A, B, C, D, E, F be 2–extensors in R^3. This means that $\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}, \overline{F}$ are 2–dimensional subspaces in R^3, or lines in $P^2(R)$. Then

$$\frac{(A \wedge B \wedge C)(D \wedge E \wedge F)}{(A \wedge B \wedge D)(C \wedge E \wedge F)}$$

is a projective invariant on these lines. Note that $A \wedge B \wedge C$, e.t.c. are 0–extensors, i.e. real numbers. By putting $F = A$, we obtain a projective invariant on five lines.

8. A projective invariant on a non-convex curve in $P^2(R)$.

We consider again a curve as in the figure below. The extensors A, B, C, c, b, D are indicated in the figure below. Then the double ratio of the points $(B \wedge C, b, B \wedge D, A \wedge B)$



is projectively invariant.

These examples demonstrate the usefulness of the presented formalism. The examples are not worked out in detail, but the interested reader should be able to complete them by employing the theory developed in the previous sections.